Chapter 4 Selected Problem Solutions

Section 4-2

4-1. a) $P(1 < X) = \int_1^\infty e^{-x}dx = (-e^{-x})\Big|_1^\infty = e^{-1} = 0.3679$

b) $P(1 < X < 2.5) = \int_1^{2.5} e^{-x}dx = (-e^{-x})\Big|_1^{2.5} = e^{-1} - e^{-2.5} = 0.2858$

c) $P(X = 3) = \int_3^3 e^{-x}dx = 0$

d) $P(X < 4) = \int_0^4 e^{-x}dx = (-e^{-x})\Big|_0^4 = 1 - e^{-4} = 0.9817$

e) $P(3 \le X) = \int_3^\infty e^{-x}dx = (-e^{-x})\Big|_3^\infty = e^{-3} = 0.0498$

4-3 a) $P(X < 4) = \int_3^4 \frac{x}{8}dx = \frac{x^2}{16}\Big|_3^4 = \frac{4^2 - 3^2}{16} = 0.4375$, because $f_X(x) = 0$ for x < 3.

b) , $P(X > 3.5) = \int_{3.5}^5 \frac{x}{8}dx = \frac{x^2}{16}\Big|_{3.5}^5 = \frac{5^2 - 3.5^2}{16} = 0.7969$ because $f_X(x) = 0$ for x > 5.

c) $P(4 < X < 5) = \int_4^5 \frac{x}{8}dx = \frac{x^2}{16}\Big|_4^5 = \frac{5^2 - 4^2}{16} = 0.5625$

d) $P(X < 4.5) = \int_3^{4.5} \frac{x}{8}dx = \frac{x^2}{16}\Big|_3^{4.5} = \frac{4.5^2 - 3^2}{16} = 0.7031$

e) $P(X > 4.5) + P(X < 3.5) = \int_{4.5}^5 \frac{x}{8}dx + \int_3^{3.5} \frac{x}{8}dx = \frac{x^2}{16}\Big|_{4.5}^5 + \frac{x^2}{16}\Big|_3^{3.5} = \frac{5^2 - 4.5^2}{16} + \frac{3.5^2 - 3^2}{16} = 0.5$.

4-9 a) P(X < 2.25 or X > 2.75) = P(X < 2.25) + P(X > 2.75) because the two events are mutually exclusive. Then, P(X < 2.25) = 0 and

P(X > 2.75) = $\int_{2.75}^{2.8} 2dx = 2(0.05) = 0.10$.

b) If the probability density function is centered at 2.5 meters, then $f_X(x) = 2$ for 2.25 < x < 2.75 and all rods will meet specifications.

Section 4-3

4-11. a) P(X<2.8) = P(X ≤ 2.8) because X is a continuous random variable.
Then, P(X<2.8) =F(2.8)=0.2(2.8) = 0.56.
b) $P(X > 1.5) = 1 - P(X \le 1.5) = 1 - 0.2(1.5) = 0.7$

c) $P(X < -2) = F_X(-2) = 0$

d) $P(X > 6) = 1 - F_X(6) = 0$

4-13. Now, $f_X(x) = e^{-x}$ for 0 < x and $F_X(x) = \int_0^x e^{-x}\,dx = -e^{-x}\Big|_0^x = 1 - e^{-x}$

for 0 < x. Then, $F_X(x) = \begin{cases} 0, x \le 0 \\ 1 - e^{-x}, x > 0 \end{cases}$

4-21. $F(x) = \int_0^x 0.5x\,dx = \frac{0.5x^2}{2}\Big|_0^x = 0.25x^2$ for 0 < x < 2. Then,

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.25x^2, & 0 \le x < 2 \\ 1, & 2 \le x \end{cases}$$

Section 4-4

4-25. $E(X) = \int_3^5 x\frac{x}{8}dx = \frac{x^3}{24}\Big|_3^5 = \frac{5^3 - 3^3}{24} = 4.083$

$$V(X) = \int_3^5 (x - 4.083)^2 \frac{x}{8}dx = \int_3^5 x^2 \tfrac{x}{8}dx - 4.083^2$$

$$= \frac{x^4}{32}\Big|_3^5 - 4.083^2 = 0.3291$$

4-27. a.) $E(X) = \int_{100}^{120} x\frac{600}{x^2}dx = 600\ln x\Big|_{100}^{120} = 109.39$

$$V(X) = \int_{100}^{120} (x - 109.39)^2 \frac{600}{x^2}dx = 600\int_{100}^{120} 1 - \tfrac{2(109.39)}{x} + \tfrac{(109.39)^2}{x^2}\,dx$$

$$= 600(x - 218.78\ln x - 109.39^2 x^{-1})\Big|_{100}^{120} = 33.19$$

b.) Average cost per part = \$0.50*109.39 = \$54.70

Section 4-5

4-33. a) f(x)= 2.0 for 49.75 < x < 50.25.
E(X) = (50.25 + 49.75)/2 = 50.0,

$$V(X) = \frac{(50.25 - 49.75)^2}{12} = 0.0208, \ \ and\ \sigma_x = 0.144.$$

b) $F(x) = \int_{49.75}^{x} 2.0dx$ for 49.75 < x < 50.25. Therefore,

$$F(x) = \begin{cases} 0, & x < 49.75 \\ 2x - 99.5, & 49.75 \le x < 50.25 \\ 1, & 50.25 \le x \end{cases}$$

c) $P(X < 50.1) = F(50.1) = 2(50.1) - 99.5 = 0.7$

4-35 $E(X) = \frac{(1.5 + 2.2)}{2} = 1.85\,\text{min}$

$$V(X) = \frac{(2.2 - 1.5)^2}{12} = 0.0408\,\text{min}^2$$

b) $P(X < 2) = \int_{1.5}^{2} \frac{1}{(2.2 - 1.5)} dx = \int_{1.5}^{2} 0.7dx = 0.7x\Big|_{1.5}^{2} = 0.7(.5) = 0.7143$

c.) $F(X) = \int_{1.5}^{x} \frac{1}{(2.2 - 1.5)} dx = \int_{1.5}^{x} 0.7dx = 0.7x\Big|_{1.5}^{x}$ for 1.5 < x < 2.2. Therefore,

$$F(x) = \begin{cases} 0, & x < 1.5 \\ 0.7x - 2.14, & 1.5 \le x < 2.2 \\ 1, & 2.2 \le x \end{cases}$$

Section 4-6

4-41 a) P(Z < 1.28) = 0.90
b) P(Z < 0) = 0.5
c) If P(Z > z) = 0.1, then P(Z < z) = 0.90 and z = 1.28
d) If P(Z > z) = 0.9, then P(Z < z) = 0.10 and z = −1.28
e) P(−1.24 < Z < z) = P(Z < z) − P(Z < −1.24)
= P(Z < z) − 0.10749.
Therefore, P(Z < z) = 0.8 + 0.10749 = 0.90749 and z = 1.33

4-43. a) P(X < 13) = P(Z < (13−10)/2)
= P(Z < 1.5)
= 0.93319

b) P(X > 9) = 1 − P(X < 9)
= 1 − P(Z < (9−10)/2)
= 1 − P(Z < −0.5)
= 1 − [1− P(Z < 0.5)]
= P(Z < 0.5)
= 0.69146.

c) P(6 < X < 14) = $P\left(\frac{6-10}{2} < Z < \frac{14-10}{2}\right)$
= P(−2 < Z < 2)
= P(Z < 2) −P(Z < − 2)]
= 0.9545.

d) P(2 < X < 4) = $P\left(\frac{2-10}{2} < Z < \frac{4-10}{2}\right)$
= P(−4 < Z < −3)
= P(Z < −3) − P(Z < −4)
= 0.00135

e) P(−2 < X < 8) = P(X < 8) − P(X < −2)
$= P\left(Z < \frac{8-10}{2}\right) - P\left(Z < \frac{-2-10}{2}\right)$
= P(Z < −1) − P(Z < −6)
= 0.15866.

4-51. a) P(X <45) = $P\left(Z < \frac{45-65}{5}\right)$
= P(Z < -3)
= 0.00135

b) P(X > 65) = $P\left(Z > \frac{65-60}{5}\right)$
= PZ >1)
= 1- P(Z < 1)
= 1 - 0.841345
= 0.158655

c) P(X < x) = $P\left(Z < \frac{x-60}{5}\right)$ = 0.99.

Therefore, $\frac{x-60}{5}$ = 2.33 and x = 71.6

4-55. a) P(X > 90.3) + P(X < 89.7)
$= P\left(Z > \frac{90.3-90.2}{0.1}\right) + P\left(Z < \frac{89.7-90.2}{0.1}\right)$
= P(Z > 1) + P(Z < −5)
= 1 − P(Z < 1) + P(Z < −5)

$=1 - 0.84134 + 0$

$= 0.15866.$

Therefore, the answer is 0.15866.

b) The process mean should be set at the center of the specifications; that is, at $\mu = 90.0$.

c) $P(89.7 < X < 90.3) = P\left(\frac{89.7 - 90}{0.1} < Z < \frac{90.3 - 90}{0.1}\right)$

$= P(-3 < Z < 3) = 0.9973.$

The yield is 100*0.9973 = 99.73%

4-59. a) $P(X > 0.0026) = P\left(Z > \frac{0.0026 - 0.002}{0.0004}\right)$

$= P(Z > 1.5)$

$= 1-P(Z < 1.5)$

$= 0.06681.$

b) $P(0.0014 < X < 0.0026) = P\left(\frac{0.0014 - 0.002}{0.0004} < Z < \frac{0.0026 - 0.002}{0.0004}\right)$

$= P(-1.5 < Z < 1.5)$

$= 0.86638.$

c) $P(0.0014 < X < 0.0026) = P\left(\frac{0.0014 - 0.002}{\sigma} < Z < \frac{0.0026 - 0.002}{\sigma}\right)$

$= P\left(\frac{-0.0006}{\sigma} < Z < \frac{0.0006}{\sigma}\right).$

Therefore, $P\left(Z < \frac{0.0006}{\sigma}\right) = 0.9975$. Therefore, $\frac{0.0006}{\sigma} = 2.81$ and $\sigma = 0.000214$.

Section 4-7

4-67 Let X denote the number of errors on a web site. Then, X is a binomial random variable with p = 0.05 and n = 100. Also, E(X) = 100 (0.05) = 5 and V(X) = 100(0.05)(0.95) = 4.75

$$P(X \geq 1) \cong P\left(Z \geq \frac{1-5}{\sqrt{4.75}}\right) = P(Z \geq -1.84) = 1 - P(Z < -1.84) = 1 - 0.03288 = 0.96712$$

4-69 Let X denote the number of hits to a web site. Then, X is a Poisson random variable with a of mean 10,000 per day. E(X) = λ = 10,000 and V(X) = 10,000

a)

$$P(X \geq 10{,}200) \cong P\left(Z \geq \frac{10{,}200 - 10{,}000}{\sqrt{10{,}000}}\right) = P(Z \geq 2) = 1 - P(Z < 2)$$

$$= 1 - 0.9772 = 0.0228$$

Expected value of hits days with more than 10,200 hits per day is
(0.0228)*365=8.32 days per year

b.) Let Y denote the number of days per year with over 10,200 hits to a web site.
Then, Y is a binomial random variable with n=365 and p=0.0228.
E(Y) = 8.32 and V(Y) = 365(0.0228)(0.9772)=8.13

$$P(Y > 15) \cong P\left(Z \geq \frac{15-8.32}{\sqrt{8.13}}\right) = P(Z \geq 2.34) = 1 - P(Z < 2.34)$$
$$= 1 - 0.9904 = 0.0096$$

Section 4-9

4-77. Let X denote the time until the first call. Then, X is exponential and
$\lambda = \frac{1}{E(X)} = \frac{1}{15}$ calls/minute.

a) $P(X > 30) = \int_{30}^{\infty} \frac{1}{15} e^{-\frac{x}{15}} dx = -e^{-\frac{x}{15}}\Big|_{30}^{\infty} = e^{-2} = 0.1353$

b) The probability of at least one call in a 10-minute interval equals one minus the probability of zero calls in
a 10-minute interval and that is P(X > 10).

$P(X > 10) = -e^{-\frac{x}{15}}\Big|_{10}^{\infty} = e^{-2/3} = 0.5134$.

Therefore, the answer is 1- 0.5134 = 0.4866. Alternatively, the requested probability is equal to P(X < 10) = 0.4866.

c) $P(5 < X < 10) = -e^{-\frac{x}{15}}\Big|_{5}^{10} = e^{-1/3} - e^{-2/3} = 0.2031$

d) P(X < x) = 0.90 and $P(X < x) = -e^{-\frac{t}{15}}\Big|_{0}^{x} = 1 - e^{-x/15} = 0.90$. Therefore, x = 34.54 minutes.

4-79. Let X denote the time to failure (in hours) of fans in a personal computer. Then, X is an exponential random variable and $\lambda = 1/E(X) = 0.0003$.

a) P(X > 10,000) = $\int_{10,000}^{\infty} 0.0003 e^{-x0.0003} dx = -e^{-x0.0003}\Big|_{10,000}^{\infty} = e^{-3} = 0.0498$

b) P(X < 7,000) = $\int_{0}^{7,000} 0.0003 e^{-x0.0003} dx = -e^{-x0.0003}\Big|_{0}^{7,000} = 1 - e^{-2.1} = 0.8775$

4-81. Let X denote the time until the arrival of a taxi. Then, X is an exponential random variable with $\lambda = 1/E(X) = 0.1$ arrivals/ minute.

a) P(X > 60) = $\int_{60}^{\infty} 0.1e^{-0.1x}dx = -e^{-0.1x}\Big|_{60}^{\infty} = e^{-6} = 0.0025$

b) P(X < 10) = $\int_{0}^{10} 0.1e^{-0.1x}dx = -e^{-0.1x}\Big|_{0}^{10} = 1 - e^{-1} = 0.6321$

4-83. Let X denote the distance between major cracks. Then, X is an exponential random variable with $\lambda = 1/E(X) = 0.2$ cracks/mile.

a) P(X > 10) = $\int_{10}^{\infty} 0.2e^{-0.2x}dx = -e^{-0.2x}\Big|_{10}^{\infty} = e^{-2} = 0.1353$

b) Let Y denote the number of cracks in 10 miles of highway. Because the distance between cracks is exponential, Y is a Poisson random variable with $\lambda = 10(0.2) = 2$ cracks per 10 miles.

$$P(Y = 2) = \frac{e^{-2}2^2}{2!} = 0.2707$$

c) $\sigma_X = 1/\lambda = 5$ miles.

4-87. Let X denote the number of calls in 3 hours. Because the time between calls is an exponential random variable, the number of calls in 3 hours is a Poisson random variable. Now, the mean time between calls is 0.5 hours and $\lambda = 1/0.5 = 2$ calls per hour = 6 calls in 3 hours.

$$P(X \geq 4) = 1 - P(X \leq 3) = 1 - \left[\frac{e^{-6}6^0}{0!} + \frac{e^{-6}6^1}{1!} + \frac{e^{-6}6^2}{2!} + \frac{e^{-6}6^3}{3!}\right] = 0.8488$$

Section 4-10

4-97. Let Y denote the number of calls in one minute. Then, Y is a Poisson random variable with $\lambda = 5$ calls per minute.

a) P(Y = 4) = $\frac{e^{-5}5^4}{4!} = 0.1755$

b) P(Y > 2) = 1 - $P(Y \leq 2) = 1 - \frac{e^{-5}5^0}{0!} - \frac{e^{-5}5^1}{1!} - \frac{e^{-5}5^2}{2!} = 0.8754$.

Let W denote the number of one minute intervals out of 10 that contain more than 2 calls. Because the calls are a Poisson process, W is a binomial random variable with n = 10 and p = 0.8754.

Therefore, P(W = 10) = $\binom{10}{10}0.8754^{10}(1-0.8754)^0 = 0.2643$.

4-101. Let X denote the number of bits until five errors occur. Then, X has an Erlang distribution with r = 5 and $\lambda = 10^{-5}$ error per bit.

a) E(X) = $\frac{r}{\lambda} = 5\times10^5$ bits.

b) V(X) = $\frac{r}{\lambda^2} = 5\times10^{10}$ and $\sigma_X = \sqrt{5\times10^{10}} = 223607$ bits.

c) Let Y denote the number of errors in 10^5 bits. Then, Y is a Poisson random variable with

$\lambda = 1/10^5 = 10^{-5}$ error per bit = 1 error per 10^5 bits.

$$P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - \left[\frac{e^{-1}1^0}{0!} + \frac{e^{-1}1^1}{1!} + \frac{e^{-1}1^2}{2!}\right] = 0.0803$$

4-105. a) $\Gamma(6) = 5! = 120$

b) $\Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\pi^{1/2} = 1.32934$

c) $\Gamma(\frac{9}{2}) = \frac{7}{2}\Gamma(\frac{7}{2}) = \frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{105}{16}\pi^{1/2} = 11.6317$

Section 4-11

4-109. β=0.2 and δ=100 hours

$$E(X) = 100\Gamma(1 + \tfrac{1}{0.2}) = 100 \times 5! = 12{,}000$$

$$V(X) = 100^2\Gamma(1 + \tfrac{2}{0.2}) - 100^2[\Gamma(1 + \tfrac{1}{0.2})]^2 = 3.61\times10^{10}$$

4-111. Let X denote lifetime of a bearing. β=2 and δ=10000 hours

a) $P(X > 8000) = 1 - F_X(8000) = e^{-\left(\frac{8000}{10000}\right)^2} = e^{-0.8^2} = 0.5273$

b)

$$E(X) = 10000\Gamma(1 + \tfrac{1}{2}) = 10000\Gamma(1.5)$$

$$= 10000(0.5)\Gamma(0.5) = 5000\sqrt{\pi} = 8862.3$$

= 8862.3 hours

c) Let Y denote the number of bearings out of 10 that last at least 8000 hours. Then, Y is a

binomial random variable with n = 10 and p = 0.5273.

$$P(Y = 10) = \binom{10}{10}0.5273^{10}(1 - 0.5273)^0 = 0.00166.$$

Section 4-12

4-117 X is a lognormal distribution with θ=5 and ω^2=9

a.)

$$P(X < 13300) = P(e^W < 13300) = P(W < \ln(13300)) = \Phi\left(\frac{\ln(13330) - 5}{3}\right)$$

$$= \Phi(1.50) = 0.9332$$

b.) Find the value for which P(X≤x)=0.95

$$P(X \le x) = P(e^W \le x) = P(W < \ln(x)) = \Phi\left(\frac{\ln(x)-5}{3}\right) = 0.95$$

$$\frac{\ln(x)-5}{3} = 1.65 \quad x = e^{1.65(3)+5} = 20952.2$$

c.) $\mu = E(X) = e^{\theta+\omega^2/2} = e^{5+9/2} = e^{9.5} = 13359.7$

$$V(X) = e^{2\theta+\omega^2}(e^{\omega^2}-1) = e^{10+9}(e^9-1) = e^{19}(e^9-1) = 1.45\times 10^{12}$$

4-119 a.) X is a lognormal distribution with θ=2 and ω^2=4

$$P(X < 500) = P(e^W < 500) = P(W < \ln(500)) = \Phi\left(\frac{\ln(500)-2}{2}\right)$$

$$= \Phi(2.11) = 0.9826$$

b.)

$$P(X < 15000 \mid X > 1000) = \frac{P(1000 < X < 1500)}{P(X > 1000)}$$

$$= \frac{\left[\Phi\left(\frac{\ln(1500)-2}{2}\right) - \Phi\left(\frac{\ln(1000)-2}{2}\right)\right]}{\left[1-\Phi\left(\frac{\ln(1000)-2}{2}\right)\right]}$$

$$= \frac{\Phi(2.66)-\Phi(2.45)}{(1-\Phi(2.45))} = \frac{0.9961-0.9929}{(1-0.9929)} = 0.0032/0.007 = 0.45$$

c.) The product has degraded over the first 1000 hours, so the probability of it lasting another 500 hours is very low.

4-121 Find the values of θand ω^2 given that E(X) = 100 and V(X) = 85,000

$x = \frac{100}{\sqrt{y}}$ $\qquad 85000 = e^{2\theta+\omega^2}(e^{\omega^2} - 1)$

let $x = e^{\theta}$ and $y = e^{\omega^2}$ then (1) $100 = x\sqrt{y}$ and (2) $85000 = x^2 y(y-1) = x^2y^2 - x^2y$

Square (1) $10000 = x^2 y$ and substitute into (2)

$85000 = 10000(y-1)$

$y = 9.5$

Substitute y into (1) and solve for x $x = \frac{100}{\sqrt{9.5}} = 32.444$

$\theta = \ln(32.444) = 3.45$ and $\omega^2 = \ln(9.5) = 2.25$

Supplemental Exercises

4-127. Let X denote the time between calls. Then, $\lambda = 1/E(X) = 0.1$ calls per minute.

a) $P(X < 5) = \int_0^5 0.1e^{-0.1x}dx = -e^{-0.1x}\Big|_0^5 = 1 - e^{-0.5} = 0.3935$

b) $P(5 < X < 15) = -e^{-0.1x}\Big|_5^{15} = e^{-0.5} - e^{-1.5} = 0.3834$

c) P(X < x) = 0.9. Then, $P(X < x) = \int_0^x 0.1e^{-0.1t}dt = 1 - e^{-0.1x} = 0.9$. Now, x = 23.03 minutes.

4-129. a) Let Y denote the number of calls in 30 minutes. Then, Y is a Poisson random variable with $x = e^{\theta}$. $\qquad P(Y \le 2) = \frac{e^{-3}3^0}{0!} + \frac{e^{-3}3^1}{1!} + \frac{e^{-3}3^2}{2!} = 0.423$.

b) Let W denote the time until the fifth call. Then, W has an Erlang distribution with $\lambda = 0.1$ and r = 5.

E(W) = 5/0.1 = 50 minutes

4-137. Let X denote the thickness.

a) P(X > 5.5) = $P\left(Z > \frac{5.5-5}{0.2}\right)$ = P(Z > 2.5) = 0. 0062

b) P(4.5 < X < 5.5) = $P\left(\frac{4.5-5}{0.2} < Z < \frac{5.5-5}{0.2}\right)$ = P (-2.5 < Z < 2.5) = 0.9876

Therefore, the proportion that do not meet specifications is 1 − P(4.5 < X < 5.5) = 0.012.

c) If P(X < x) = 0.90, then $P\left(Z > \frac{x-5}{0.2}\right)$ = 0.9. Therefore, $\frac{x-5}{0.2}$ = 1.65 and x = 5.33.

4-139. If P(0.002-x < X < 0.002+x), then P(-x/0.0004 < Z < x/0.0004) = 0.9973. Therefore, x/0.0004 = 3 and x = 0.0012. The specifications are from 0.0008 to 0.0032.

4-141. If P(X > 10,000) = 0.99, then P(Z > $\frac{10{,}000-\mu}{600}$) = 0.99. Therefore, $\frac{10{,}000-\mu}{600}$ = -2.33 and $\mu = 11{,}398$.

4-143 X is an exponential distribution with E(X) = 7000 hours

a.) $P(X < 5800) = \int_0^{5800} \frac{1}{7000} e^{-\frac{x}{7000}} dx = 1 - e^{-\left(\frac{5800}{7000}\right)} = 0.5633$

b.) $P(X > x) = \int_x^{\infty} \frac{1}{7000} e^{-\frac{x}{7000}} dx = 0.9$ Therefore, $e^{-\frac{x}{7000}} = 0.9$

and $x = -7000\ln(0.9) = 737.5$ hours