Chapter 7 Selected Problem Solutions

Section 7-2

7-7. $E(\hat{\Theta}_1) = \theta$ No bias $V(\hat{\Theta}_1) = 12 = MSE(\hat{\Theta}_1)$

$E(\hat{\Theta}_2) = \theta$ No bias $V(\hat{\Theta}_2) = 10 = MSE(\hat{\Theta}_2)$

$E(\hat{\Theta}_3) \neq \theta$ Bias $MSE(\hat{\Theta}_3) = 6$ [not that this includes (bias^2)]

To compare the three estimators, calculate the relative efficiencies:

$\frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)} = \frac{12}{10} = 1.2$, since rel. eff. > 1 use $\hat{\Theta}_2$ as the estimator for θ

$\frac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_3)} = \frac{12}{6} = 2$, since rel. eff. > 1 use $\hat{\Theta}_3$ as the estimator for θ

$\frac{MSE(\hat{\Theta}_2)}{MSE(\hat{\Theta}_3)} = \frac{10}{6} = 1.8$, since rel. eff. > 1 use $\hat{\Theta}_3$ as the estimator for θ

Conclusion:

$\hat{\Theta}_3$ is the most efficient estimator with bias, but it is biased. $\hat{\Theta}_2$ is the best "unbiased" estimator.

7-11

a.) The average of the 26 observations provided can be used as an estimator of the mean pull force since we know it is unbiased. This value is 75.427 pounds.
b.) The median of the sample can be used as an estimate of the point that divides the population into a "weak" and "strong" half. This estimate is 75.1 pounds.
c.) Our estimate of the population variance is the sample variance or 2.214 square pounds. Similarly, our estimate of the population standard deviation is the sample standard deviation or 1.488 pounds.
d.) The standard error of the mean pull force, estimated from the data provided is 0.292 pounds. This value is the standard deviation, not of the pull force, but of the mean pull force of the population.
e.) Only one connector in the sample has a pull force measurement under 73 pounds. Our point estimate for the proportion requested is then 1/26 = 0.0385

7-13 a.) To see if the estimator is unbiased, find:

$$E[(X_{\min} + X_{\max})/2] = \frac{1}{2}[E(X_{\min}) + E(X_{\max})] = \frac{1}{2}(\mu + \mu) = \mu$$

since the expected value of any observation arising from a normally distributed process is equal to the mean. So this is an unbiased estimator of the mean.

b.) The standard error of this estimator is:

$$\sqrt{V[(X_{\min} + X_{\max})/2]} = \frac{1}{2}\sqrt{[V(X_{\min}) + V(X_{\max}) + COV(X_{\min}, X_{\max})]} = \frac{1}{2}\sqrt{(\sigma^2 + \sigma^2)} = \frac{1}{\sqrt{2}}\sigma$$

c.) This estimator is not better than the sample mean because it has larger standard error for n > 2. This is due to the fact that this estimator uses only two observations from the available sample. The sample mean uses all the information available to compute the estimate.

7-17

a) $E(\hat{\mu}) = E(\alpha\overline{X}_1 + (1-\alpha)\overline{X}_2) = \alpha E(\overline{X}_1) + (1-\alpha)E(\overline{X}_2) = \alpha\mu + (1-\alpha)\mu = \mu$

b)

$$s.e.(\hat{\mu}) = \sqrt{V(\alpha\overline{X}_1 + (1-\alpha)\overline{X}_2)} = \sqrt{\alpha^2 V(\overline{X}_1) + (1-\alpha)^2 V(\overline{X}_2)}$$
$$= \sqrt{\alpha^2 \frac{\sigma_1^{\ 2}}{n_1} + (1-\alpha)^2 \frac{\sigma_2^{\ 2}}{n_2}} = \sqrt{\alpha^2 \frac{\sigma_1^{\ 2}}{n_1} + (1-\alpha)^2 a \frac{\sigma_1^{\ 2}}{n_2}}$$
$$= \sigma_1 \sqrt{\frac{\alpha^2 n_2 + (1-\alpha)^2 a n_1}{n_1 n_2}}$$

c) The value of alpha that minimizes the standard error is:

$$\alpha = \frac{an_1}{n_2 + an_1}$$

d) With $a = 4$ and $n_1=2n_2$, the value of alpha to choose is 8/9. The arbitrary value of $\alpha=0.5$ is too small and will result in a larger standard error. With $\alpha=8/9$ the standard error is

$$s.e.(\hat{\mu}) = \sigma_1 \sqrt{\frac{(8/9)^2 n_2 + (1/9)^2 8n_2}{2n_2^{\ 2}}} = \frac{0.667\sigma_1}{\sqrt{n_2}}$$

If $\alpha=0.05$ the standard error is

$$s.e.(\hat{\mu}) = \sigma_1 \sqrt{\frac{(0.5)^2 n_2 + (0.5)^2 8n_2}{2n_2^{\ 2}}} = \frac{1.0607\sigma_1}{\sqrt{n_2}}$$

Section 7-5

7-33. $$P(1.009 \le \overline{X} \le 1.012) = P\left(\frac{1.009 - 1.01}{0.003/\sqrt{9}} \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le \frac{1.012 - 1.01}{0.003/\sqrt{9}}\right)$$
$$= P(-1 \le Z \le 2) = P(Z \le 2) - P(Z \le -1)$$
$$= 0.9772 - 0.1587 = 0.8385$$

7-35. $$\mu_{\overline{X}} = 75.5\,psi\,,\ \sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{3.5}{\sqrt{6}} = 1.429$$
$$P(\overline{X} \ge 75.75) = P\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \ge \frac{75.75-75.5}{1.429}\right)$$
$$= P(Z \ge 0.175) = 1 - P(Z \le 1.75)$$
$$= 1 - 0.56945 = 0.43055$$

7-39 $\sigma^2 = 25$

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}$$

$$n = \left(\frac{\sigma}{\sigma_{\bar{X}}}\right)^2 = \left(\frac{5}{1.5}\right)^2 = 11.11$$

$n \cong 12$

7-41 n = 36

$$\mu_X = \frac{a+b}{2} = \frac{(3+1)}{2} = 2$$

$$\sigma_X = \sqrt{\frac{(b-a+1)^2-1}{12}} = \sqrt{\frac{(3-1+1)^2-1}{12}} = \sqrt{\frac{8}{12}} = \sqrt{\frac{2}{3}}$$

$$\mu_{\bar{X}} = 2, \sigma_{\bar{X}} = \frac{\sqrt{2/3}}{\sqrt{36}} = \frac{\sqrt{2/3}}{6}$$

$$z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$

Using the central limit theorem:

$$P(2.1 < \bar{X} < 2.5) = P\left(\frac{2.1-2}{\frac{\sqrt{2/3}}{6}} < Z < \frac{2.5-2}{\frac{\sqrt{2/3}}{6}}\right)$$

$$= P(0.7348 < Z < 3.6742)$$

$$= P(Z < 3.6742) - P(Z < 0.7348)$$

$$= 1 - 0.7688 = 0.2312$$

7-43.

$n_1 = 16$	$n_2 = 9$	$\bar{X}_1 - \bar{X}_2 \sim N(\mu_{\bar{X}_1} - \mu_{\bar{X}_2}, \sigma^2_{\bar{X}_1} + \sigma^2_{\bar{X}_2})$
$\mu_1 = 75$	$\mu_2 = 70$	
$\sigma_1 = 8$	$\sigma_2 = 12$	$\sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$
		$\sim N(75-70, \frac{8^2}{16} + \frac{12^2}{9})$
		$\sim N(5,20)$

a) $P(\bar{X}_1 - \bar{X}_2 > 4)$

$$P(Z > \tfrac{4-5}{\sqrt{20}}) = P(Z > -0.2236) = 1 - P(Z \le -0.2236)$$

$$= 1 - 0.4115 = 0.5885$$

b) $P(3.5 \le \overline{X}_1 - \overline{X}_2 \le 5.5)$

$$P(\tfrac{3.5-5}{\sqrt{20}} \le Z \le \tfrac{5.5-5}{\sqrt{20}}) = P(Z \le 0.1118) - P(Z \le -0.3354)$$

$$= 0.5445 - 0.3686 = 0.1759$$

Supplemental Exercises

7-49. $\overline{X}_1 - \overline{X}_2 \sim N(100-105, \frac{1.5^2}{25} + \frac{2^2}{25}) \sim N(-5, 0.2233)$