2 Probability

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Understand and describe sample spaces and events for random experiments with graphs, tables, lists, or tree diagrams
2. Interpret probabilities and use probabilities of outcomes to calculate probabilities of events in discrete sample spaces
3. Calculate the probabilities of joint events such as unions and intersections from the probabilities of individual events
4. Interpret and calculate conditional probabilities of events
5. Determine the independence of events and use independence to calculate probabilities
6. Use Bayes' theorem to calculate conditional probabilities
7. Understand random variables

CD MATERIAL

8. Use permutation and combinations to count the number of outcomes in both an event and the sample space.

Answers for most odd numbered exercises are at the end of the book. Answers to exercises whose numbers are surrounded by a box can be accessed in the e-Text by clicking on the box. Complete worked solutions to certain exercises are also available in the e-Text. These are indicated in the Answers to Selected Exercises section by a box around the exercise number. Exercises are also available for some of the text sections that appear on CD only. These exercises may be found within the e-Text immediately following the section they accompany.

2-1 SAMPLE SPACES AND EVENTS

2-1.1 Random Experiments

If we measure the current in a thin copper wire, we are conducting an experiment. However, in day-to-day repetitions of the measurement the results can differ slightly because of small variations in variables that are not controlled in our experiment, including changes in ambient temperatures, slight variations in gauge and small impurities in the chemical composition of the wire if different locations are selected, and current source drifts. Consequently, this experiment (as well as many we conduct) is said to have a **random** component. In some cases, the random variations, are small enough, relative to our experimental goals, that they can be ignored. However, no matter how carefully our experiment is designed and conducted, the variation is almost always present, and its magnitude can be large enough that the important conclusions from our experiment are not obvious. In these cases, the methods presented in this book for modeling and analyzing experimental results are quite valuable.

Our goal is to understand, quantify, and model the type of variations that we often encounter. When we incorporate the variation into our thinking and analyses, we can make informed judgments from our results that are not invalidated by the variation.

Models and analyses that include variation are not different from models used in other areas of engineering and science. Figure 2-1 displays the important components. A mathematical model (or abstraction) of the physical system is developed. It need not be a perfect abstraction. For example, Newton's laws are not perfect descriptions of our physical universe. Still, they are useful models that can be studied and analyzed to approximately quantify the performance of a wide range of engineered products. Given a mathematical abstraction that is validated with measurements from our system, we can use the model to understand, describe, and quantify important aspects of the physical system and predict the response of the system to inputs.

Throughout this text, we discuss models that allow for variations in the outputs of a system, even though the variables that we control are not purposely changed during our study. Figure 2-2 graphically displays a model that incorporates uncontrollable inputs (noise) that combine with the controllable inputs to produce the output of our system. Because of the

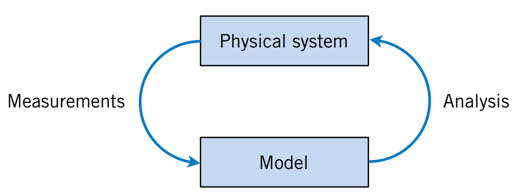


Figure 2-1 Continuous iteration between model and physical system.

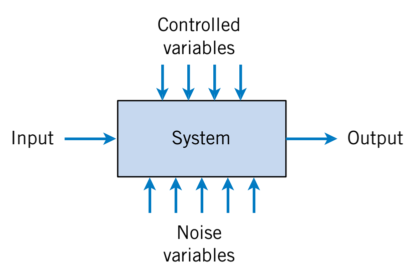


Figure 2-2 Noise variables affect the transformation of inputs to outputs.

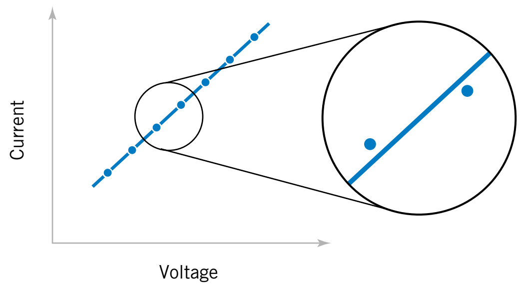


Figure 2-3 A closer examination of the system identifies deviations from the model.

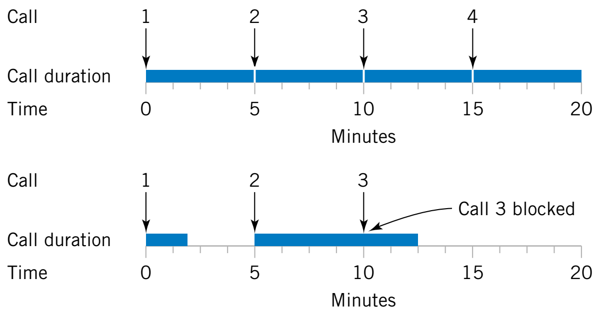


Figure 2-4 Variation causes disruptions in the system.

uncontrollable inputs, the same settings for the controllable inputs do not result in identical outputs every time the system is measured.

Definition

An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a **random experiment.**

For the example of measuring current in a copper wire, our model for the system might simply be Ohm's law. Because of uncontrollable inputs, variations in measurements of current are expected. Ohm's law might be a suitable approximation. However, if the variations are large relative to the intended use of the device under study, we might need to extend our model to include the variation. See Fig. 2-3.

As another example, in the design of a communication system, such as a computer or voice communication network, the information capacity available to service individuals using the network is an important design consideration. For voice communication, sufficient external lines need to be purchased from the phone company to meet the requirements of a business. Assuming each line can carry only a single conversation, how many lines should be purchased? If too few lines are purchased, calls can be delayed or lost. The purchase of too many lines increases costs. Increasingly, design and product development is required to meet customer requirements *at a competitive cost.*

In the design of the voice communication system, a model is needed for the number of calls and the duration of calls. Even knowing that on average, calls occur every five minutes and that they last five minutes is not sufficient. If calls arrived precisely at five-minute intervals and lasted for precisely five minutes, one phone line would be sufficient. However, the slightest variation in call number or duration would result in some calls being blocked by others. See Fig. 2-4. A system designed without considering variation will be woefully inadequate for practical use. Our model for the number and duration of calls needs to include variation as an integral component. An analysis of models including variation is important for the design of the phone system.

2-1.2 Sample Spaces

To model and analyze a random experiment, we must understand the set of possible **outcomes** from the experiment. In this introduction to probability, we make use of the basic concepts of sets and operations on sets. It is assumed that the reader is familiar with these topics.

Definition

> The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S.

A sample space is often defined based on the objectives of the analysis.

EXAMPLE 2-1

Consider an experiment in which you select a molded plastic part, such as a connector, and measure its thickness. The possible values for thickness depend on the resolution of the measuring instrument, and they also depend on upper and lower bounds for thickness. However, it might be convenient to define the sample space as simply the positive real line

$$S = R^+ = \{x \mid x > 0\}$$

because a negative value for thickness cannot occur.

If it is known that all connectors will be between 10 and 11 millimeters thick, the sample space could be

$$S = \{x \mid 10 < x < 11\}$$

If the objective of the analysis is to consider only whether a particular part is low, medium, or high for thickness, the sample space might be taken to be the set of three outcomes:

$$S = \{low, medium, high\}$$

If the objective of the analysis is to consider only whether or not a particular part conforms to the manufacturing specifications, the sample space might be simplified to the set of two outcomes

$$S = \{yes, no\}$$

that indicate whether or not the part conforms.

It is useful to distinguish between two types of sample spaces.

Definition

> A sample space is **discrete** if it consists of a finite or countable infinite set of outcomes.
> A sample space is **continuous** if it contains an interval (either finite or infinite) of real numbers.

In Example 2-1, the choice $S = R^+$ is an example of a continuous sample space, whereas $S = \{yes, no\}$ is a discrete sample space. As mentioned, the best choice of a sample space depends on the objectives of the study. As specific questions occur later in the book, appropriate sample spaces are discussed.

EXAMPLE 2-2 If two connectors are selected and measured, the extension of the positive real line R is to take the sample space to be the positive quadrant of the plane:

$$S = R^+ \times R^+$$

If the objective of the analysis is to consider only whether or not the parts conform to the manufacturing specifications, either part may or may not conform. We abbreviate *yes* and *no* as y and n. If the ordered pair yn indicates that the first connector conforms and the second does not, the sample space can be represented by the four outcomes:

$$S = \{yy, yn, ny, nn\}$$

If we are only interested in the number of conforming parts in the sample, we might summarize the sample space as

$$S = \{0, 1, 2\}$$

As another example, consider an experiment in which the thickness is measured until a connector fails to meet the specifications. The sample space can be represented as

$$S = \{n, yn, yyn, yyyn, yyyyn, \text{and so forth}\}$$

In random experiments in which items are selected from a batch, we will indicate whether or not a selected item is replaced before the next one is selected. For example, if the batch consists of three items $\{a, b, c\}$ and our experiment is to select two items **without replacement,** the sample space can be represented as

$$S_{\text{without}} = \{ab, ac, ba, bc, ca, cb\}$$

This description of the sample space maintains the order of the items selected so that the outcome *ab* and *ba* are separate elements in the sample space. A sample space with less detail only describes the two items selected $\{\{a, b\}, \{a, c\}, \{b, c\}\}$. This sample space is the possible subsets of two items. Sometimes the ordered outcomes are needed, but in other cases the simpler, unordered sample space is sufficient.

If items are replaced before the next one is selected, the sampling is referred to as **with replacement.** Then the possible ordered outcomes are

$$S_{\text{with}} = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$$

The unordered description of the sample space is $\{\{a, a\}, \{a, b\}, \{a, c\}, \{b, b\}, \{b, c\}, \{c, c\}\}$. Sampling without replacement is more common for industrial applications.

Sometimes it is not necessary to specify the exact item selected, but only a property of the item. For example, suppose that there are 5 defective parts and 95 good parts in a batch. To study the quality of the batch, two are selected without replacement. Let g denote a good part and d denote a defective part. It might be sufficient to describe the sample space (ordered) in terms of quality of each part selected as

$$S = \{gg, gd, dg, dd\}$$

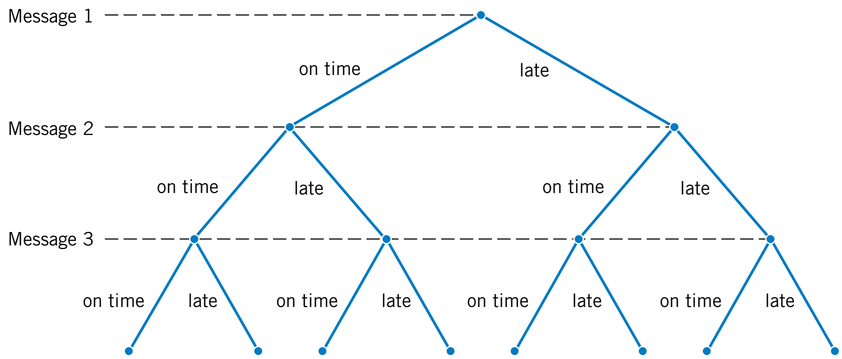


Figure 2-5 Tree diagram for three messages.

One must be cautious with this description of the sample space because there are many more pairs of items in which both are good than pairs in which both are defective. These differences must be accounted for when probabilities are computed later in this chapter. Still, this summary of the sample space will be convenient when conditional probabilities are used later in this chapter. Also, if there were only one defective part in the batch, there would be fewer possible outcomes

$$S = \{gg, gd, dg\}$$

because *dd* would be impossible. For sampling questions, sometimes the most important part of the solution is an appropriate description of the sample space.

Sample spaces can also be described graphically with **tree diagrams.** When a sample space can be constructed in several steps or stages, we can represent each of the n_1 ways of completing the first step as a branch of a tree. Each of the ways of completing the second step can be represented as n_2 branches starting from the ends of the original branches, and so forth.

EXAMPLE 2-3

Each message in a digital communication system is classified as to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.

Each message can either be received on time or late. The possible results for three messages can be displayed by eight branches in the tree diagram shown in Fig. 2-5.

EXAMPLE 2-4

An automobile manufacturer provides vehicles equipped with selected options. Each vehicle is ordered

With or without an automatic transmission

With or without air-conditioning

With one of three choices of a stereo system

With one of four exterior colors

If the sample space consists of the set of all possible vehicle types, what is the number of outcomes in the sample space? The sample space contains 48 outcomes. The tree diagram for the different types of vehicles is displayed in Fig. 2-6.

EXAMPLE 2-5

Consider an extension of the automobile manufacturer illustration in the previous example in which another vehicle option is the interior color. There are four choices of interior color: red, black, blue, or brown. However,

With a red exterior, only a black or red interior can be chosen.

With a white exterior, any interior color can be chosen.

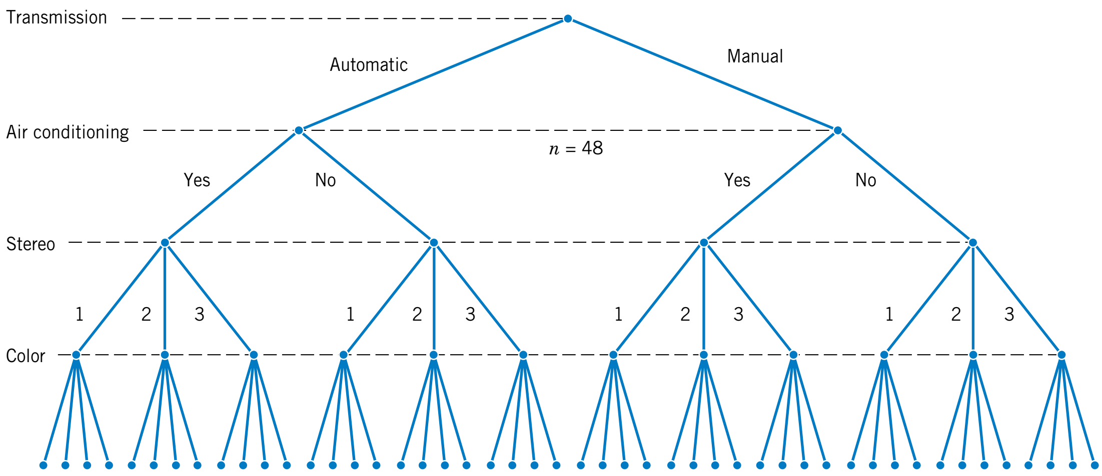


Figure 2-6 Tree diagram for different types of vehicles.

With a blue exterior, only a black, red, or blue interior can be chosen.

With a brown exterior, only a brown interior can be chosen.

In Fig. 2-6, there are 12 vehicle types with each exterior color, but the number of interior color choices depends on the exterior color. As shown in Fig. 2-7, the tree diagram can be extended to show that there are 120 different vehicle types in the sample space.

2-1.3 Events

Often we are interested in a collection of related outcomes from a random experiment.

Definition

An **event** is a subset of the sample space of a random experiment.

We can also be interested in describing new events from combinations of existing events. Because events are subsets, we can use basic set operations such as unions, intersections, and

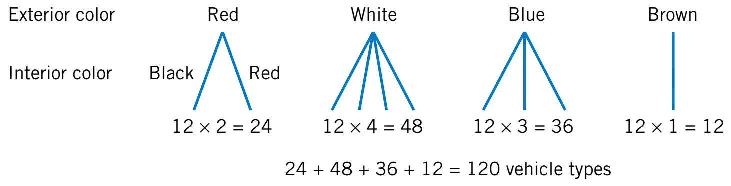


Figure 2-7 Tree diagram for different types of vehicles with interior colors.

complements to form other events of interest. Some of the basic set operations are summarized below in terms of events:

- The **union** of two events is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.
- The **intersection** of two events is the event that consists of all outcomes that are contained in both of the two events. We denote the intersection as $E_1 \cap E_2$.
- The **complement** of an event in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event E as E'.

EXAMPLE 2-6

Consider the sample space $S = \{yy, yn, ny, nn\}$ in Example 2-2. Suppose that the set of all outcomes for which at least one part conforms is denoted as E_1. Then,

$$E_1 = \{yy, yn, ny\}$$

The event in which both parts do not conform, denoted as E_2, contains only the single outcome, $E_2 = \{nn\}$. Other examples of events are $E_3 = \emptyset$, the null set, and $E_4 = S$, the sample space. If $E_5 = \{yn, ny, nn\}$,

$$E_1 \cup E_5 = S \qquad E_1 \cap E_5 = \{yn, ny\} \qquad E_1' = \{nn\}$$

EXAMPLE 2-7

Measurements of the time needed to complete a chemical reaction might be modeled with the sample space $S = R^+$, the set of positive real numbers. Let

$$E_1 = \{x \mid 1 \leq x < 10\} \quad \text{and} \quad E_2 = \{x \mid 3 < x < 118\}$$

Then,

$$E_1 \cup E_2 = \{x \mid 1 \leq x < 118\} \quad \text{and} \quad E_1 \cap E_2 = \{x \mid 3 < x < 10\}$$

Also,

$$E_1' = \{x \mid x \geq 10\} \quad \text{and} \quad E_1' \cap E_2 = \{x \mid 10 \leq x < 118\}$$

EXAMPLE 2-8

Samples of polycarbonate plastic are analyzed for scratch and shock resistance. The results from 50 samples are summarized as follows:

		shock resistance	
		high	low
scratch resistance	high	40	4
	low	1	5

Let A denote the event that a sample has high shock resistance, and let B denote the event that a sample has high scratch resistance. Determine the number of samples in $A \cap B$, A', and $A \cup B$.

The event $A \cap B$ consists of the 40 samples for which scratch and shock resistances are high. The event A' consists of the 9 samples in which the shock resistance is low. The event $A \cup B$ consists of the 45 samples in which the shock resistance, scratch resistance, or both are high.

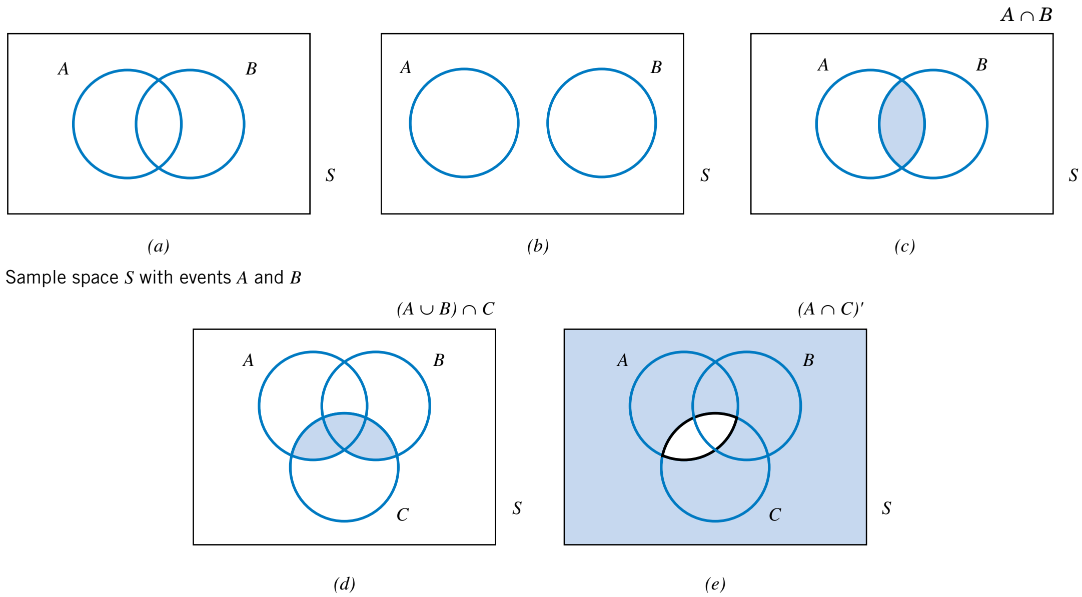


Figure 2-8 Venn diagrams.

Diagrams are often used to portray relationships between sets, and these diagrams are also used to describe relationships between events. We can use **Venn diagrams** to represent a sample space and events in a sample space. For example, in Fig. 2-8(a) the sample space of the random experiment is represented as the points in the rectangle S. The events A and B are the subsets of points in the indicated regions. Figure 2-8(b) illustrates two events with no common outcomes; Figs. 2-8(c) to 2-8(e) illustrate additional joint events.

Two events with no outcomes in common have an important relationship.

Definition

Two events, denoted as E_1 and E_2, such that

$$E_1 \cap E_2 = \varnothing$$

are said to be **mutually exclusive.**

The two events in Fig. 2-8(b) are mutually exclusive, whereas the two events in Fig. 2-8(a) are not.

Additional results involving events are summarized below. The definition of the complement of an event implies that

$$(E')' = E$$

The distributive law for set operations implies that

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \qquad \text{and} \qquad (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

DeMorgan's laws imply that

$$(A \cup B)' = A' \cap B' \quad \text{and} \quad (A \cap B)' = A' \cup B'$$

Also, remember that

$$A \cap B = B \cap A \quad \text{and} \quad A \cup B = B \cup A$$

2-1.4 Counting Techniques (CD Only)

As sample spaces become larger, complete enumeration is difficult. Instead, counts of the number outcomes in the sample space and in various events are often used to analyze the random experiment. These methods are referred to as **counting techniques** and described on the CD.

EXERCISES FOR SECTION 2-1

Provide a reasonable description of the sample space for each of the random experiments in Exercises 2-1 to 2-18. There can be more than one acceptable interpretation of each experiment. Describe any assumptions you make.

2-1. Each of three machined parts is classified as either above or below the target specification for the part.

2-2. Each of four transmitted bits is classified as either in error or not in error.

2-3. In the final inspection of electronic power supplies, three types of nonconformities might occur: functional, minor, or cosmetic. Power supplies that are defective are further classified as to type of nonconformity.

2-4. In the manufacturing of digital recording tape, electronic testing is used to record the number of bits in error in a 350-foot reel.

2-5. In the manufacturing of digital recording tape, each of 24 tracks is classified as containing or not containing one or more bits in error.

2-6. An ammeter that displays three digits is used to measure current in milliamperes.

2-7. A scale that displays two decimal places is used to measure material feeds in a chemical plant in tons.

2-8. The following two questions appear on an employee survey questionnaire. Each answer is chosen from the five-point scale 1 (never), 2, 3, 4, 5 (always).

Is the corporation willing to listen to and fairly evaluate new ideas?

How often are my coworkers important in my overall job performance?

2-9. The concentration of ozone to the nearest part per billion.

2-10. The time until a tranaction service is requested of a computer to the nearest millisecond.

2-11. The pH reading of a water sample to the nearest tenth of a unit.

2-12. The voids in a ferrite slab are classified as small, medium, or large. The number of voids in each category is measured by an optical inspection of a sample.

2-13. The time of a chemical reaction is recorded to the nearest millisecond.

2-14. An order for an automobile can specify either an automatic or a standard transmission, either with or without air-conditioning, and any one of the four colors red, blue, black or white. Describe the set of possible orders for this experiment.

2-15. A sampled injection-molded part could have been produced in either one of two presses and in any one of the eight cavities in each press.

2-16. An order for a computer system can specify memory of 4, 8, or 12 gigabytes, and disk storage of 200, 300, or 400 gigabytes. Describe the set of possible orders.

2-17. Calls are repeatedly placed to a busy phone line until a connect is achieved.

2-18. In a magnetic storage device, three attempts are made to read data before an error recovery procedure that repositions the magnetic head is used. The error recovery procedure attempts three repositionings before an "abort" message is sent to the operator. Let

s denote the success of a read operation

f denote the failure of a read operation

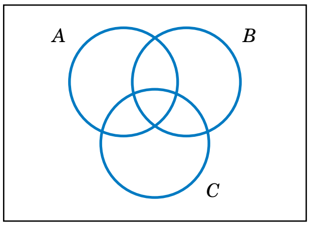
F denote the failure of an error recovery procedure

S denote the success of an error recovery procedure

A denote an abort message sent to the operator.

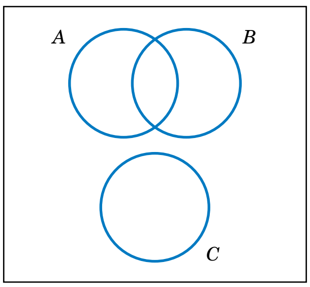
Describe the sample space of this experiment with a tree diagram.

2-19. Three events are shown on the Venn diagram in the following figure:



Reproduce the figure and shade the region that corresponds to each of the following events.
(a) A' (b) $A \cap B$
(c) $(A \cap B) \cup C$ (d) $(B \cup C)'$
(e) $(A \cap B)' \cup C$

2-20. Three events are shown on the Venn diagram in the following figure:



Reproduce the figure and shade the region that corresponds to each of the following events.
(a) A' (b) $(A \cap B) \cup (A \cap B')$
(c) $(A \cap B) \cup C$ (d) $(B \cup C)'$
(e) $(A \cap B)' \cup C$

2-21. A digital scale is used that provides weights to the nearest gram.
(a) What is the sample space for this experiment?
Let A denote the event that a weight exceeds 11 grams, let B denote the event that a weight is less than or equal to 15 grams, and let C denote the event that a weight is greater than or equal to 8 grams and less than 12 grams.
Describe the following events.
(b) $A \cup B$ (c) $A \cap B$
(d) A' (e) $A \cup B \cup C$
(f) $(A \cup C)'$ (g) $A \cap B \cap C$
(h) $B' \cap C$ (i) $A \cup (B \cap C)$

2-22. In an injection-molding operation, the length and width, denoted as X and Y, respectively, of each molded part are evaluated. Let

A denote the event of $48 < X < 52$ centimeters
B denote the event of $9 < Y < 11$ centimeters
C denote the event that a critical length meets customer requirements.

Construct a Venn diagram that includes these events. Shade the areas that represent the following:
(a) A (b) $A \cap B$
(c) $A' \cup B$ (d) $A \cup B$
(e) If these events were mutually exclusive, how successful would this production operation be? Would the process produce parts with $X = 50$ centimeters and $Y = 10$ centimeters?

2-23. Four bits are transmitted over a digital communications channel. Each bit is either distorted or received without distortion. Let A_i denote the event that the ith bit is distorted, $i = 1, \ldots, 4$.
(a) Describe the sample space for this experiment.
(b) Are the A_i's mutually exclusive?
Describe the outcomes in each of the following events:
(c) A_1 (d) A_1'
(e) $A_1 \cap A_2 \cap A_3 \cap A_4$ (f) $(A_1 \cap A_2) \cup (A_3 \cap A_4)$

2-24. A sample of three calculators is selected from a manufacturing line, and each calculator is classified as either defective or acceptable. Let A, B, and C denote the events that the first, second, and third calculators respectively, are defective.
(a) Describe the sample space for this experiment with a tree diagram.
Use the tree diagram to describe each of the following events:
(b) A (c) B
(d) $A \cap B$ (e) $B \cup C$

2-25. A wireless garage door opener has a code determined by the up or down setting of 12 switches. How many outcomes are in the sample space of possible codes?

2-26. Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized below:

		shock resistance	
		high	low
scratch	high	70	9
resistance	low	16	5

Let A denote the event that a disk has high shock resistance, and let B denote the event that a disk has high scratch resistance. Determine the number of disks in $A \cap B$, A', and $A \cup B$.

2-27. Samples of a cast aluminum part are classified on the basis of surface finish (in microinches) and edge finish. The results of 100 parts are summarized as follows:

		edge finish	
		excellent	good
surface	excellent	80	2
finish	good	10	8

(a) Let A denote the event that a sample has excellent surface finish, and let B denote the event that a sample has excellent edge finish. Determine the number of samples in $A' \cap B$, B', and $A \cup B$.
(b) Assume that each of two samples is to be classified on the basis of surface finish, either excellent or good, edge finish, either excellent or good. Use a tree diagram to represent the possible outcomes of this experiment.

2-28. Samples of emissions from three suppliers are classified for conformance to air-quality specifications. The results from 100 samples are summarized as follows:

		conforms	
		yes	no
	1	22	8
supplier	2	25	5
	3	30	10

Let A denote the event that a sample is from supplier 1, and let B denote the event that a sample conforms to specifications. Determine the number of samples in $A' \cap B$, B', and $A \cup B$.

2-29. The rise time of a reactor is measured in minutes (and fractions of minutes). Let the sample space be positive, real numbers. Define the events A and B as follows:

$$A = \{x \mid x < 72.5\}$$

and

$$B = \{x \mid x > 52.5\}$$

Describe each of the following events:
(a) A' (b) B'
(c) $A \cap B$ (d) $A \cup B$

2-30. A sample of two items is selected without replacement from a batch. Describe the (ordered) sample space for each of the following batches:
(a) The batch contains the items $\{a, b, c, d\}$.
(b) The batch contains the items $\{a, b, c, d, e, f, g\}$.
(c) The batch contains 4 defective items and 20 good items.
(d) The batch contains 1 defective item and 20 good items.

2-31. A sample of two printed circuit boards is selected without replacement from a batch. Describe the (ordered) sample space for each of the following batches:
(a) The batch contains 90 boards that are not defective, 8 boards with minor defects, and 2 boards with major defects.
(b) The batch contains 90 boards that are not defective, 8 boards with minor defects, and 1 board with major defects.

2-32. Counts of the Web pages provided by each of two computer servers in a selected hour of the day are recorded. Let A denote the event that at least 10 pages are provided by server 1 and let B denote the event that at least 20 pages are provided by server 2.
(a) Describe the sample space for the numbers of pages for two servers graphically.

Show each of the following events on the sample space graph:
(b) A (c) B
(d) $A \cap B$ (e) $A \cup B$

2-33. The rise time of a reactor is measured in minutes (and fractions of minutes). Let the sample space for the rise time of each batch be positive, real numbers. Consider the rise times of *two* batches. Let A denote the event that the rise time of batch 1 is less than 72.5 minutes, and let B denote the event that the rise time of batch 2 is greater than 52.5 minutes.

Describe the sample space for the rise time of two batches graphically and show each of the following events on a two-dimensional plot:
(a) A (b) B'
(c) $A \cap B$ (d) $A \cup B$

2-2 INTERPRETATIONS OF PROBABILITY

2-2.1 Introduction

In this chapter, we introduce probability for **discrete sample spaces**—those with only a finite (or countably infinite) set of outcomes. The restriction to these sample spaces enables us to simplify the concepts and the presentation without excessive mathematics.

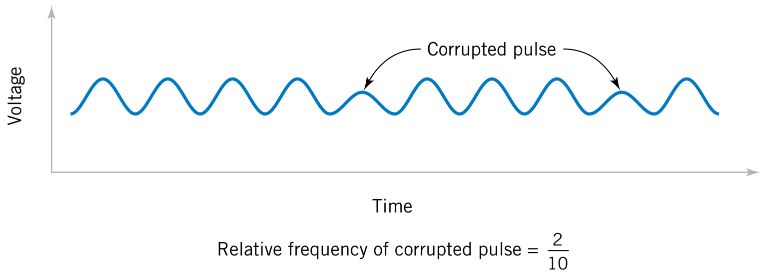


Figure 2-9 Relative frequency of corrupted pulses sent over a communication channel.

Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur. "The chance of rain today is 30%" is a statement that quantifies our feeling about the possibility of rain. The likelihood of an outcome is quantified by assigning a number from the interval [0, 1] to the outcome (or a percentage from 0 to 100%). Higher numbers indicate that the outcome is more likely than lower numbers. A 0 indicates an outcome will not occur. A probability of 1 indicates an outcome will occur with certainty.

The probability of an outcome can be interpreted as our subjective probability, or **degree of belief,** that the outcome will occur. Different individuals will no doubt assign different probabilities to the same outcomes. Another interpretation of probability is based on the conceptual model of repeated replications of the random experiment. The probability of an outcome is interpreted as the limiting value of the proportion of times the outcome occurs in n repetitions of the random experiment as n increases beyond all bounds. For example, if we assign probability 0.2 to the outcome that there is a corrupted pulse in a digital signal, we might interpret this assignment as implying that, if we analyze many pulses, approximately 20% of them will be corrupted. This example provides a **relative frequency** interpretation of probability. The proportion, or relative frequency, of replications of the experiment that result in the outcome is 0.2. Probabilities are chosen so that the sum of the probabilities of all outcomes in an experiment add up to 1. This convention facilitates the relative frequency interpretation of probability. Figure 2-9 illustrates the concept of relative frequency.

Probabilities for a random experiment are often assigned on the basis of a reasonable model of the system under study. One approach is to base probability assignments on the simple concept of equally likely outcomes.

For example, suppose that we will select one laser diode **randomly** from a batch of 100. The sample space is the set of 100 diodes. *Randomly* implies that it is reasonable to assume that each diode in the batch has an equal chance of being selected. Because the sum of the probabilities must equal 1, the probability model for this experiment assigns probability of 0.01 to each of the 100 outcomes. We can interpret the probability by imagining many replications of the experiment. Each time we start with all 100 diodes and select one at random. The probability 0.01 assigned to a particular diode represents the proportion of replicates in which a particular diode is selected.

When the model of **equally likely outcomes** is assumed, the probabilities are chosen to be equal.

> Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

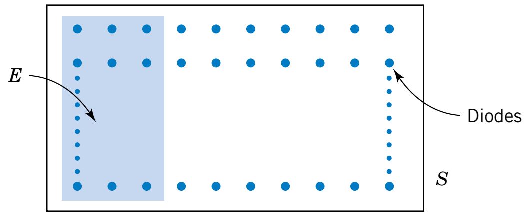


Figure 2-10 Probability of the event E is the sum of the probabilities of the outcomes in E.

It is frequently necessary to assign probabilities to events that are composed of several outcomes from the sample space. This is straightforward for a discrete sample space.

EXAMPLE 2-9

Assume that 30% of the laser diodes in a batch of 100 meet the minimum power requirements of a specific customer. If a laser diode is selected randomly, that is, each laser diode is equally likely to be selected, our intuitive feeling is that the probability of meeting the customer's requirements is 0.30.

Let E denote the subset of 30 diodes that meet the customer's requirements. Because E contains 30 outcomes and each outcome has probability 0.01, we conclude that the probability of E is 0.3. The conclusion matches our intuition. Figure 2-10 illustrates this example.

For a discrete sample space, the probability of an event can be defined by the reasoning used in the example above.

Definition

For a discrete sample space, the *probability of an event E,* denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

EXAMPLE 2-10

A random experiment can result in one of the outcomes $\{a, b, c, d\}$ with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event $\{a, b\}$, B the event $\{b, c, d\}$, and C the event $\{d\}$.Then,

$$P(A) = 0.1 + 0.3 = 0.4$$
$$P(B) = 0.3 + 0.5 + 0.1 = 0.9$$
$$P(C) = 0.1$$

Also, $P(A') = 0.6$, $P(B') = 0.1$, and $P(C') = 0.9$. Furthermore, because $A \cap B = \{b\}$, $P(A \cap B) = 0.3$. Because $A \cup B = \{a, b, c, d\}$, $P(A \cup B) = 0.1 + 0.3 + 0.5 + 0.1 = 1$. Because $A \cap C$ is the null set, $P(A \cap C) = 0$.

EXAMPLE 2-11

A visual inspection of a location on wafers from a semiconductor manufacturing process resulted in the following table:

Number of Contamination Particles	Proportion of Wafers
0	0.40
1	0.20
2	0.15
3	0.10
4	0.05
5 or more	0.10

If one wafer is selected randomly from this process and the location is inspected, what is the probability that it contains no particles? If information were available for each wafer, we could define the sample space as the set of all wafers inspected and proceed as in the example with diodes. However, this level of detail is not needed in this case. We can consider the sample space to consist of the six categories that summarize the number of contamination particles on a wafer. Then, the event that there is no particle in the inspected location on the wafer, denoted as E, can be considered to be comprised of the single outcome, namely, $E = \{0\}$. Therefore,

$$P(E) = 0.4$$

What is the probability that a wafer contains three or more particles in the inspected location? Let E denote the event that a wafer contains three or more particles in the inspected location. Then, E consists of the three outcomes $\{3, 4, 5 \text{ or more}\}$. Therefore,

$$P(E) = 0.10 + 0.05 + 0.10 = 0.25$$

EXAMPLE 2-12

Suppose that a batch contains six parts with part numbers $\{a, b, c, d, e, f\}$. Suppose that two parts are selected without replacement. Let E denote the event that the part number of the first part selected is a. Then E can be written as $E = \{ab, ac, ad, ae, af\}$. The sample space can be enumerated. It has 30 outcomes. If each outcome is equally likely, $P(E) = 5/30 = 1/6$.

Also, if E_2 denotes the event that the second part selected is a, $E_2 = \{ba, ca, da, ea, fa\}$ and with equally likely outcomes, $P(E_2) = 5/30 = 1/6$.

2-2.2 Axioms of Probability

Now that the probability of an event has been defined, we can collect the assumptions that we have made concerning probabilities into a set of **axioms** that the probabilities in any random experiment must satisfy. The axioms ensure that the probabilities assigned in an experiment can be interpreted as relative frequencies and that the assignments are consistent with our intuitive understanding of relationships between relative frequencies. For example, if event A is contained in event B, we should have $P(A) \leq P(B)$. The **axioms do not determine probabilities;** the probabilities are assigned based on our knowledge of the system under study. However, the axioms enable us to easily calculate the probabilities of some events from knowledge of the probabilities of other events.

Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

If S is the sample space and E is any event in a random experiment,

(1) $P(S) = 1$

(2) $0 \leq P(E) \leq 1$

(3) For two events E_1 and E_2 with $E_1 \cap E_2 = \varnothing$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

The property that $0 \leq P(E) \leq 1$ is equivalent to the requirement that a relative frequency must be between 0 and 1. The property that $P(S) = 1$ is a consequence of the fact that an outcome from the sample space occurs on every trial of an experiment. Consequently, the relative frequency of S is 1. Property 3 implies that if the events E_1 and E_2 have no outcomes in common, the relative frequency of outcomes in $E_1 \cup E_2$ is the sum of the relative frequencies of the outcomes in E_1 and E_2.

These axioms imply the following results. The derivations are left as exercises at the end of this section. Now,

$$P(\varnothing) = 0$$

and for any event E,

$$P(E') = 1 - P(E)$$

For example, if the probability of the event E is 0.4, our interpretation of relative frequency implies that the probability of E' is 0.6. Furthermore, if the event E_1 is contained in the event E_2,

$$P(E_1) \leq P(E_2)$$

EXERCISES FOR SECTION 2-2

2-34. Each of the possible five outcomes of a random experiment is equally likely. The sample space is $\{a, b, c, d, e\}$. Let A denote the event $\{a, b\}$, and let B denote the event $\{c, d, e\}$. Determine the following:
(a) $P(A)$ (b) $P(B)$
(c) $P(A')$ (d) $P(A \cup B)$
(e) $P(A \cap B)$

2-35. The sample space of a random experiment is $\{a, b, c, d, e\}$ with probabilities 0.1, 0.1, 0.2, 0.4, and 0.2, respectively. Let A denote the event $\{a, b, c\}$, and let B denote the event $\{c, d, e\}$. Determine the following:
(a) $P(A)$ (b) $P(B)$
(c) $P(A')$ (d) $P(A \cup B)$
(e) $P(A \cap B)$

2-36. A part selected for testing is equally likely to have been produced on any one of six cutting tools.
(a) What is the sample space?
(b) What is the probability that the part is from tool 1?
(c) What is the probability that the part is from tool 3 or tool 5?
(d) What is the probability that the part is not from tool 4?

2-37. An injection-molded part is equally likely to be obtained from any one of the eight cavities on a mold.
(a) What is the sample space?
(b) What is the probability a part is from cavity 1 or 2?
(c) What is the probability that a part is neither from cavity 3 nor 4?

2-38. A sample space contains 20 equally likely outcomes. If the probability of event A is 0.3, how many outcomes are in event A?

2-39. Orders for a computer are summarized by the optional features that are requested as follows:

	proportion of orders
no optional features	0.3
one optional feature	0.5
more than one optional feature	0.2

(a) What is the probability that an order requests at least one optional feature?
(b) What is the probability that an order does not request more than one optional feature?

2-40. If the last digit of a weight measurement is equally likely to be any of the digits 0 through 9,
(a) What is the probability that the last digit is 0?
(b) What is the probability that the last digit is greater than or equal to 5?

2-41. A sample preparation for a chemical measurement is completed correctly by 25% of the lab technicians, completed with a minor error by 70%, and completed with a major error by 5%.
(a) If a technician is selected randomly to complete the preparation, what is the probability it is completed without error?
(b) What is the probability that it is completed with either a minor or a major error?

2-42. A credit card contains 16 digits between 0 and 9. However, only 100 million numbers are valid. If a number is entered randomly, what is the probability that it is a valid number?

2-43. Suppose your vehicle is licensed in a state that issues license plates that consist of three digits (between 0 and 9) followed by three letters (between A and Z). If a license number is selected randomly, what is the probability that yours is the one selected?

2-44. A message can follow different paths through servers on a network. The senders message can go to one of five servers for the first step, each of them can send to five servers at the second step, each of which can send to four servers at the third step, and then the message goes to the recipients server.
(a) How many paths are possible?
(b) If all paths are equally likely, what is the probability that a message passes through the first of four servers at the third step?

2-45. Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

		shock resistance	
		high	low
scratch	high	70	9
resistance	low	16	5

Let A denote the event that a disk has high shock resistance, and let B denote the event that a disk has high scratch resistance. If a disk is selected at random, determine the following probabilities:
(a) $P(A)$ (b) $P(B)$
(c) $P(A')$ (d) $P(A \cap B)$
(e) $P(A \cup B)$ (f) $P(A' \cup B)$

2-46. Samples of a cast aluminum part are classified on the basis of surface finish (in microinches) and edge finish. The results of 100 parts are summarized as follows:

		edge finish	
		excellent	good
surface	excellent	80	2
finish	good	10	8

Let A denote the event that a sample has excellent surface finish, and let B denote the event that a sample has excellent length. If a part is selected at random, determine the following probabilities:
(a) $P(A)$ (b) $P(B)$
(c) $P(A')$ (d) $P(A \cap B)$
(e) $P(A \cup B)$ (f) $P(A' \cup B)$

2-47. Samples of emissions from three suppliers are classified for conformance to air-quality specifications. The results from 100 samples are summarized as follows:

		conforms	
		yes	no
	1	22	8
supplier	2	25	5
	3	30	10

Let A denote the event that a sample is from supplier 1, and let B denote the event that a sample conforms to specifications. If a sample is selected at random, determine the following probabilities:
(a) $P(A)$ (b) $P(B)$
(c) $P(A')$ (d) $P(A \cap B)$
(e) $P(A \cup B)$ (f) $P(A' \cup B)$

2-48. Use the axioms of probability to show the following:
(a) For any event E, $P(E') = 1 - P(E)$.
(b) $P(\varnothing) = 0$
(c) If A is contained in B, then $P(A) \leq P(B)$

2-3 ADDITION RULES

Joint events are generated by applying basic set operations to individual events. Unions of events, such as $A \cup B$; intersections of events, such as $A \cap B$; and complements of events, such as A', are commonly of interest. The probability of a joint event can often be determined from the probabilities of the individual events that comprise it. Basic set operations are also sometimes helpful in determining the probability of a joint event. In this section the focus is on unions of events.

EXAMPLE 2-13 Table 2-1 lists the history of 940 wafers in a semiconductor manufacturing process. Suppose one wafer is selected at random. Let H denote the event that the wafer contains high levels of contamination. Then, $P(H) = 358/940$.

Let C denote the event that the wafer is in the center of a sputtering tool. Then, $P(C) = 626/940$. Also, $P(H \cap C)$ is the probability that the wafer is from the center of the sputtering tool and contains high levels of contamination. Therefore,

$$P(H \cap C) = 112/940$$

The event $H \cup C$ is the event that a wafer is from the center of the sputtering tool or contains high levels of contamination (or both). From the table, $P(H \cup C) = 872/940$. An alternative calculation of $P(H \cup C)$ can be obtained as follows. The 112 wafers that comprise the event $H \cap C$ are included once in the calculation of $P(H)$ and again in the calculation of $P(C)$. Therefore, $P(H \cup C)$ can be found to be

$$\begin{aligned} P(H \cup C) &= P(H) + P(C) - P(H \cap C) \\ &= 358/940 + 626/940 - 112/940 = 872/940 \end{aligned}$$

The preceding example illustrates that the probability of A or B is interpreted as $P(A \cup B)$ and that the following general **addition rule** applies.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{2-1}$$

EXAMPLE 2-14 The wafers such as those described in Example 2-13 were further classified as either in the "center" or at the "edge" of the sputtering tool that was used in manufacturing, and by the degree of contamination. Table 2-2 shows the proportion of wafers in each category. What is

Table 2-1 Wafers in Semiconductor Manufacturing Classified by Contamination and Location

	Location in Sputtering Tool		
Contamination	Center	Edge	Total
Low	514	68	582
High	112	246	358
Total	626	314	

Table 2-2 Wafers Classified by Contamination and Location

Number of Contamination Particles	Center	Edge	Totals
0	0.30	0.10	0.40
1	0.15	0.05	0.20
2	0.10	0.05	0.15
3	0.06	0.04	0.10
4	0.04	0.01	0.05
5 or more	0.07	0.03	0.10
Totals	0.72	0.28	1.00

the probability that a wafer was either at the edge or that it contains four or more particles? Let E_1 denote the event that a wafer contains four or more particles, and let E_2 denote the event that a wafer is at the edge.

The requested probability is $P(E_1 \cup E_2)$. Now, $P(E_1) = 0.15$ and $P(E_2) = 0.28$. Also, from the table, $P(E_1 \cap E_2) = 0.04$. Therefore, using Equation 2-1, we find that

$$P(E_1 \cup E_2) = 0.15 + 0.28 - 0.04 = 0.39$$

What is the probability that a wafer contains less than two particles or that it is both at the edge and contains more than four particles? Let E_1 denote the event that a wafer contains less than two particles, and let E_2 denote the event that a wafer is both from the edge and contains more than four particles. The requested probability is $P(E_1 \cup E_2)$. Now, $P(E_1) = 0.60$ and $P(E_2) = 0.03$. Also, E_1 and E_2 are mutually exclusive. Consequently, there are no wafers in the intersection and $P(E_1 \cap E_2) = 0$. Therefore,

$$P(E_1 \cup E_2) = 0.60 + 0.03 = 0.63$$

Recall that two events A and B are said to be mutually exclusive if $A \cap B = \varnothing$. Then, $P(A \cap B) = 0$, and the general result for the probability of $A \cup B$ simplifies to the third axiom of probability.

If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B) \qquad (2\text{-}2)$$

Three or More Events

More complicated probabilities, such as $P(A \cup B \cup C)$, can be determined by repeated use of Equation 2-1 and by using some basic set operations. For example,

$$P(A \cup B \cup C) = P[(A \cup B) \cup C] = P(A \cup B) + P(C) - P[(A \cup B) \cap C]$$

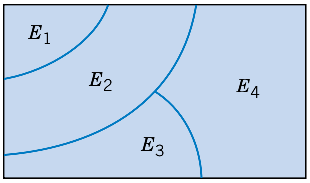


Figure 2-11 Venn diagram of four mutually exclusive events.

Upon expanding $P(A \cup B)$ by Equation 2-1 and using the distributed rule for set operations to simplify $P[(A \cup B) \cap C]$, we obtain

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) - P(A \cap B) + P(C) - P[(A \cap C) \cup (B \cap C)] \\ &= P(A) + P(B) - P(A \cap B) + P(C) \\ &\qquad - [P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)] \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) \\ &\qquad - P(B \cap C) + P(A \cap B \cap C) \end{aligned}$$

We have developed a formula for the probability of the union of three events. Formulas can be developed for the probability of the union of any number of events, although the formulas become very complex. As a summary, for the case of three events

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) \\ - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \end{aligned} \quad (2\text{-}3)$$

Results for three or more events simplify considerably if the events are mutually exclusive. In general, a collection of events, $E_1, E_2, \ldots, E_k$, is said to be mutually exclusive if there is no overlap among any of them.

The Venn diagram for several mutually exclusive events is shown in Fig. 2-11. By generalizing the reasoning for the union of two events, the following result can be obtained:

A collection of events, $E_1, E_2, \ldots, E_k$, is said to be **mutually exclusive** if for all pairs,

$$E_i \cap E_j = \varnothing$$

For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup \ldots \cup E_k) = P(E_1) + P(E_2) + \ldots P(E_k) \quad (2\text{-}4)$$

EXAMPLE 2-15 A simple example of mutually exclusive events will be used quite frequently. Let X denote the pH of a sample. Consider the event that X is greater than 6.5 but less than or equal to 7.8. This probability is the sum of any collection of mutually exclusive events with union equal to the same range for X. One example is

$$P(6.5 < X \leq 7.8) = P(6.5 < X \leq 7.0) + P(7.0 < X \leq 7.5) + P(7.5 < X \leq 7.8)$$

Another example is

$$P(6.5 < X \leq 7.8) = P(6.5 < X \leq 6.6) + P(6.6 < X \leq 7.1) + P(7.1 < X \leq 7.4) + P(7.4 < X \leq 7.8)$$

The best choice depends on the particular probabilities available.

EXERCISES FOR SECTION 2-3

2-49. If $P(A) = 0.3$, $P(B) = 0.2$, and $P(A \cap B) = 0.1$, determine the following probabilities:

(a) $P(A')$ (b) $P(A \cup B)$
(c) $P(A' \cap B)$ (d) $P(A \cap B')$
(e) $P[(A \cup B)']$ (f) $P(A' \cup B)$

2-50. If A, B, and C are mutually exclusive events with $P(A) = 0.2$, $P(B) = 0.3$, and $P(C) = 0.4$, determine the following probabilities:

(a) $P(A \cup B \cup C)$ (b) $P(A \cap B \cap C)$
(c) $P(A \cap B)$ (d) $P[(A \cup B) \cap C]$
(e) $P(A' \cap B' \cap C')$

2-51. If A, B, and C are mutually exclusive events, is it possible for $P(A) = 0.3$, $P(B) = 0.4$, and $P(C) = 0.5$? Why or why not?

2-52. Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

		shock resistance	
		high	low
scratch	high	70	9
resistance	low	16	5

(a) If a disk is selected at random, what is the probability that its scratch resistance is high and its shock resistance is high?
(b) If a disk is selected at random, what is the probability that its scratch resistance is high or its shock resistance is high?
(c) Consider the event that a disk has high scratch resistance and the event that a disk has high shock resistance. Are these two events mutually exclusive?

2-53. The analysis of shafts for a compressor is summarized by conformance to specifications.

		roundness conforms	
		yes	no
surface finish	yes	345	5
conforms	no	12	8

(a) If a shaft is selected at random, what is the probability that the shaft conforms to surface finish requirements?
(b) What is the probability that the selected shaft conforms to surface finish requirements or to roundness requirements?
(c) What is the probability that the selected shaft either conforms to surface finish requirements or does not conform to roundness requirements?
(d) What is the probability that the selected shaft conforms to both surface finish and roundness requirements?

2-54. Cooking oil is produced in two main varieties: mono- and polyunsaturated. Two common sources of cooking oil are corn and canola. The following table shows the number of bottles of these oils at a supermarket:

		type of oil	
		canola	corn
type of	mono	7	13
unsaturation	poly	93	77

(a) If a bottle of oil is selected at random, what is the probability that it belongs to the polyunsaturated category?
(b) What is the probability that the chosen bottle is monounsaturated canola oil?

2-55. A manufacturer of front lights for automobiles tests lamps under a high humidity, high temperature environment using intensity and useful life as the responses of interest. The following table shows the performance of 130 lamps:

		useful life	
		satisfactory	unsatisfactory
intensity	satisfactory	117	3
	unsatisfactory	8	2

(a) Find the probability that a randomly selected lamp will yield unsatisfactory results under any criteria.
(b) The customers for these lamps demand 95% satisfactory results. Can the lamp manufacturer meet this demand?

2-56. The shafts in Exercise 2-53 are further classified in terms of the machine tool that was used for manufacturing the shaft.

Tool 1

		roundness conforms	
		yes	no
surface finish	yes	200	1
conforms	no	4	2

Tool 2

		roundness conforms	
		yes	no
surface finish	yes	145	4
conforms	no	8	6

(a) If a shaft is selected at random, what is the probability that the shaft conforms to surface finish requirements or to roundness requirements or is from Tool 1?

(b) If a shaft is selected at random, what is the probability that the shaft conforms to surface finish requirements or does not conform to roundness requirements or is from Tool 2?

(c) If a shaft is selected at random, what is the probability that the shaft conforms to both surface finish and roundness requirements or the shaft is from Tool 2?

(d) If a shaft is selected at random, what is the probability that the shaft conforms to surface finish requirements or the shaft is from Tool 2?

2-4 CONDITIONAL PROBABILITY

A digital communication channel has an error rate of one bit per every thousand transmitted. Errors are rare, but when they occur, they tend to occur in bursts that affect many consecutive bits. If a single bit is transmitted, we might model the probability of an error as $1/1000$. However, if the previous bit was in error, because of the bursts, we might believe that the probability that the next bit is in error is greater than $1/1000$.

In a thin film manufacturing process, the proportion of parts that are not acceptable is 2%. However, the process is sensitive to contamination problems that can increase the rate of parts that are not acceptable. If we knew that during a particular shift there were problems with the filters used to control contamination, we would assess the probability of a part being unacceptable as higher than 2%.

In a manufacturing process, 10% of the parts contain visible surface flaws and 25% of the parts with surface flaws are (functionally) defective parts. However, only 5% of parts without surface flaws are defective parts. The probability of a defective part depends on our knowledge of the presence or absence of a surface flaw.

These examples illustrate that probabilities need to be reevaluated as additional information becomes available. The notation and details are further illustrated for this example.

Let D denote the event that a part is defective and let F denote the event that a part has a surface flaw. Then, we denote the probability of D given, or assuming, that a part has a surface flaw as $P(D|F)$. This notation is read as the **conditional probability** of D given F, and it is interpreted as the probability that a part is defective, given that the part has a surface flaw. Because 25% of the parts with surface flaws are defective, our conclusion can be stated as $P(D|F) = 0.25$. Furthermore, because F' denotes the event that a part does not have a surface flaw and because 5% of the parts without surface flaws are defective, we have that $P(D|F') = 0.05$. These results are shown graphically in Fig. 2-12.

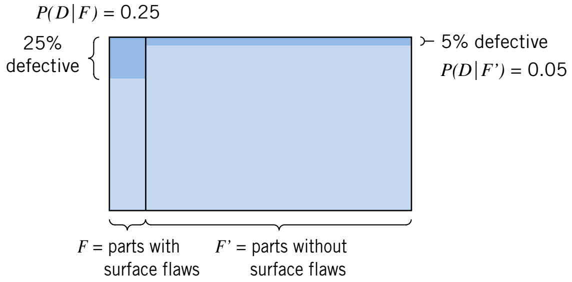


Figure 2-12 Conditional probabilities for parts with surface flaws.

Table 2-3 Parts Classified

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	38
	No	30	342	362
	Total	40	360	400

EXAMPLE 2-16

Table 2-3 provides an example of 400 parts classified by surface flaws and as (functionally) defective. For this table the conditional probabilities match those discussed previously in this section. For example, of the parts with surface flaws (40 parts) the number defective is 10. Therefore,

$$P(D|F) = 10/40 = 0.25$$

and of the parts without surface flaws (360 parts) the number defective is 18. Therefore,

$$P(D|F') = 18/360 = 0.05$$

In Example 2-16 conditional probabilities were calculated directly. These probabilities can also be determined from the formal definition of conditional probability.

Definition

The **conditional probability** of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = P(A \cap B)/P(A) \tag{2-5}$$

for $P(A) > 0$.

This definition can be understood in a special case in which all outcomes of a random experiment are equally likely. If there are n total outcomes,

$$P(A) = (\text{number of outcomes in } A)/n$$

Also,

$$P(A \cap B) = (\text{number of outcomes in } A \cap B)/n$$

Consequently,

$$P(A \cap B)/P(A) = \frac{\text{number of outcomes in } A \cap B}{\text{number of outcomes in } A}$$

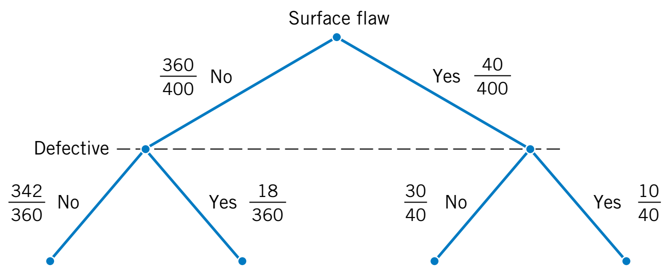


Figure 2-13 Tree diagram for parts classified

Therefore, $P(B|A)$ can be interpreted as the relative frequency of event B among the trials that produce an outcome in event A.

EXAMPLE 2-17

Again consider the 400 parts in Table 2-3. From this table

$$P(D|F) = P(D \cap F)/P(F) = \frac{10}{400}\bigg/\frac{40}{400} = \frac{10}{40}$$

Note that in this example all four of the following probabilities are different:

$$P(F) = 40/400 \qquad P(F|D) = 10/28$$
$$P(D) = 28/400 \qquad P(D|F) = 10/40$$

Here, $P(D)$ and $P(D|F)$ are probabilities of the same event, but they are computed under two different states of knowledge. Similarly, $P(F)$ and $P(F|D)$ are computed under two different states of knowledge.

The tree diagram in Fig. 2-13 can also be used to display conditional probabilities. The first branch is on surface flaw. Of the 40 parts with surface flaws, 10 are functionally defective and 30 are not. Therefore,

$$P(D|F) = 10/40 \qquad \text{and} \qquad P(D'|F) = 30/40$$

Of the 360 parts without surface flaws, 18 are functionally defective and 342 are not. Therefore,

$$P(D|F') = 342/360 \qquad \text{and} \qquad P(D'|F') = 18/360$$

Random Samples from a Batch

Recall that to select one item randomly from a batch implies that each item is equally likely. If more than one item is selected, *randomly* implies that each element of the sample space is equally likely. For example, when sample spaces were presented earlier in this chapter, sampling with and without replacement were defined and illustrated for the simple case of a batch with three items $\{a, b, c\}$. If two items are selected randomly from this batch without replacement, each of the six outcomes in the ordered sample space

$$S_{\text{without}} = \{ab, ac, ba, bc, ca, cb\}$$

has probability 1/6. If the unordered sample space is used, each of the three outcomes in $\{\{a, b\}, \{a, c\}, \{b, c\}\}$ has probability 1/3.

What is the conditional probability that b is selected second given that a is selected first? Because this question considers the results of each pick, the ordered sample space is used. The definition of conditional probability is applied as follows. Let E_1 denote the event that the first item selected is a and let E_2 denote the event that the second item selected is b. Then,

$$E_1 = \{ab, ac\} \qquad \text{and} \qquad E_2 = \{ab, cb\} \qquad \text{and} \qquad E_1 \cap E_2 = \{ab\}$$

and from the definition of conditional probability

$$P(E_2 | E_1) = P(E_1 \cap E_2)/P(E_1) = \frac{1/6}{1/3} = 1/2$$

When the sample space is larger, an alternative calculation is usually more convenient. For example, suppose that a batch contains 10 parts from tool 1 and 40 parts from tool 2. If two parts are selected randomly, without replacement, what is the conditional probability that a part from tool 2 is selected second given that a part from tool 1 is selected first? There are 50 possible parts to select first and 49 to select second. Therefore, the (ordered) sample space has $50 \times 49 = 2450$ outcomes. Let E_1 denote the event that the first part is from tool 1 and E_2 denote the event that the second part is from tool 2. As above, a count of the number of outcomes in E_1 and the intersection is needed.

Although the answer can be determined from this start, this type of question can be answered more easily with the following result.

> To select randomly implies that at each step of the sample, the items that remain in the batch are equally likely to be selected.

If a part from tool 1 were selected with the first pick, 49 items would remain, 9 from tool 1 and 40 from tool 2, and they would be equally likely to be picked. Therefore, the probability that a part from tool 2 would be selected with the second pick given this first pick is

$$P(E_2 | E_1) = 40/49.$$

In this manner, other probabilities can also be simplified. For example, let the event E consist of the outcomes with the first selected part from tool 1 and the second part from tool 2. To determine the probability of E, consider each step. The probability that a part from tool 1 is selected with the first pick is $P(E_1) = 10/50$. The conditional probability that a part from tool 2 is selected with the second pick, given that a part from tool 1 is selected first is $P(E_2 | E_1) = 40/49$. Therefore,

$$P(E) = P(E_2 | E_1)P(E_1) = \frac{40}{49} \cdot \frac{10}{50} = 0.163$$

Sometimes a partition of the question into successive picks is an easier method to solve the problem.

EXAMPLE 2-18 A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected randomly without replacement from the batch. What is the probability that the second part is defective given that the first part is defective?

Let A denote the event that the first part selected is defective, and let B denote the event that the second part selected is defective. The probability needed can be expressed as $P(B|A)$. If the first part is defective, prior to selecting the second part, the batch contains 849 parts, of which 49 are defective, therefore

$$P(B|A) = 49/849$$

EXAMPLE 2-19 Continuing the previous example, if three parts are selected at random, what is the probability that the first two are defective and the third is not defective? This event can be described in shorthand notation as simply $P(ddn)$. We have

$$P(ddn) = \frac{50}{850} \cdot \frac{49}{849} \cdot \frac{800}{848} = 0.0032$$

The third term is obtained as follows. After the first two parts are selected, there are 848 remaining. Of the remaining parts, 800 are not defective. In this example, it is easy to obtain the solution with a conditional probability for each selection.

EXERCISES FOR SECTION 2-4

2-57. Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

		shock resistance	
		high	low
scratch	high	70	9
resistance	low	16	5

Let A denote the event that a disk has high shock resistance, and let B denote the event that a disk has high scratch resistance. Determine the following probabilities:
(a) $P(A)$ (b) $P(B)$
(c) $P(A|B)$ (d) $P(B|A)$

2-58. Samples of a cast aluminum part are classified on the basis of surface finish (in microinches) and length measurements. The results of 100 parts are summarized as follows:

		length	
		excellent	good
surface	excellent	80	2
finish	good	10	8

Let A denote the event that a sample has excellent surface finish, and let B denote the event that a sample has excellent length. Determine:
(a) $P(A)$ (b) $P(B)$
(c) $P(A|B)$ (d) $P(B|A)$
(e) If the selected part has excellent surface finish, what is the probability that the length is excellent?
(f) If the selected part has good length, what is the probability that the surface finish is excellent?

2-59. The analysis of shafts for a compressor is summarized by conformance to specifications:

		roundness conforms	
		yes	no
surface finish	yes	345	5
conforms	no	12	8

(a) If we know that a shaft conforms to roundness requirements, what is the probability that it conforms to surface finish requirements?
(b) If we know that a shaft does not conform to roundness requirements, what is the probability that it conforms to surface finish requirements?

2-60. The following table summarizes the analysis of samples of galvanized steel for coating weight and surface roughness:

		coating weight	
		high	low
surface	high	12	16
roughness	low	88	34

(a) If the coating weight of a sample is high, what is the probability that the surface roughness is high?
(b) If the surface roughness of a sample is high, what is the probability that the coating weight is high?
(c) If the surface roughness of a sample is low, what is the probability that the coating weight is low?

2-61. Consider the data on wafer contamination and location in the sputtering tool shown in Table 2-2. Assume that one wafer is selected at random from this set. Let A denote the event that a wafer contains four or more particles, and let B denote the event that a wafer is from the center of the sputtering tool. Determine:
(a) $P(A)$ (b) $P(A|B)$
(c) $P(B)$ (d) $P(B|A)$
(e) $P(A \cap B)$ (f) $P(A \cup B)$

2-62. A lot of 100 semiconductor chips contains 20 that are defective. Two are selected randomly, without replacement, from the lot.
(a) What is the probability that the first one selected is defective?
(b) What is the probability that the second one selected is defective given that the first one was defective?
(c) What is the probability that both are defective?
(d) How does the answer to part (b) change if chips selected were replaced prior to the next selection?

2-63. A lot contains 15 castings from a local supplier and 25 castings from a supplier in the next state. Two castings are selected randomly, without replacement, from the lot of 40. Let A be the event that the first casting selected is from the local supplier, and let B denote the event that the second casting is selected from the local supplier. Determine:
(a) $P(A)$ (b) $P(B|A)$
(c) $P(A \cap B)$ (d) $P(A \cup B)$

2-64. Continuation of Exercise 2-63. Suppose three castings are selected at random, without replacement, from the lot of 40. In addition to the definitions of events A and B, let C denote the event that the third casting selected is from the local supplier. Determine:
(a) $P(A \cap B \cap C)$
(b) $P(A \cap B \cap C')$

2-65. A batch of 500 containers for frozen orange juice contains 5 that are defective. Two are selected, at random, without replacement from the batch.
(a) What is the probability that the second one selected is defective given that the first one was defective?
(b) What is the probability that both are defective?
(c) What is the probability that both are acceptable?

2-66. Continuation of Exercise 2-65. Three containers are selected, at random, without replacement, from the batch.
(a) What is the probability that the third one selected is defective given that the first and second one selected were defective?
(b) What is the probability that the third one selected is defective given that the first one selected was defective and the second one selected was okay?
(c) What is the probability that all three are defective?

2-67. A maintenance firm has gathered the following information regarding the failure mechanisms for air conditioning systems:

		evidence of gas leaks	
		yes	no
evidence of	yes	55	17
electrical failure	no	32	3

The units without evidence of gas leaks or electrical failure showed other types of failure. If this is a representative sample of AC failure, find the probability
(a) That failure involves a gas leak
(b) That there is evidence of electrical failure given that there was a gas leak
(c) That there is evidence of a gas leak given that there is evidence of electrical failure

2-68. If $P(A|B) = 1$, must $A = B$? Draw a Venn diagram to explain your answer.

2-69. Suppose A and B are mutually exclusive events. Construct a Venn diagram that contains the three events A, B, and C such that $P(A|C) = 1$ and $P(B|C) = 0$?

2-5 MULTIPLICATION AND TOTAL PROBABILITY RULES

2-5.1 Multiplication Rule

The definition of conditional probability in Equation 2-5 can be rewritten to provide a general expression for the probability of the intersection of two events. This formula is referred to as a **multiplication rule** for probabilities.

Multiplication Rule

$$P(A \cap B) = P(B|A)P(A) = P(A|B)P(B) \qquad (2\text{-}6)$$

The last expression in Equation 2-6 is obtained by interchanging A and B.

EXAMPLE 2-20 The probability that an automobile battery subject to high engine compartment temperature suffers low charging current is 0.7. The probability that a battery is subject to high engine compartment temperature is 0.05.

Let C denote the event that a battery suffers low charging current, and let T denote the event that a battery is subject to high engine compartment temperature. The probability that a battery is subject to low charging current and high engine compartment temperature is

$$P(C \cap T) = P(C|T)P(T) = 0.7 \times 0.05 = 0.035$$

2-5.2 Total Probability Rule

The multiplication rule is useful for determining the probability of an event that depends on other events. For example, suppose that in semiconductor manufacturing the probability is 0.10 that a chip that is subjected to high levels of contamination during manufacturing causes a product failure. The probability is 0.005 that a chip that is not subjected to high contamination levels during manufacturing causes a product failure. In a particular production run, 20% of the chips are subject to high levels of contamination. What is the probability that a product using one of these chips fails?

Clearly, the requested probability depends on whether or not the chip was exposed to high levels of contamination. We can solve this problem by the following reasoning. For any event B, we can write B as the union of the part of B in A and the part of B in A'. That is,

$$B = (A \cap B) \cup (A' \cap B)$$

This result is shown in the Venn diagram in Fig. 2-14. Because A and A' are mutually exclusive, $A \cap B$ and $A' \cap B$ are mutually exclusive. Therefore, from the probability of the union of mutually exclusive events in Equation 2-2 and the Multiplication Rule in Equation 2-6, the following **total probability rule** is obtained.

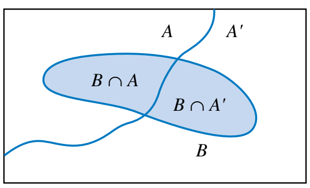
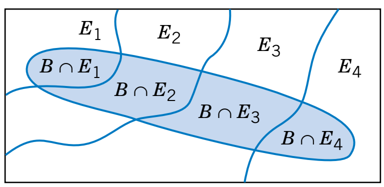


Figure 2-14 Partitioning an event into two mutually exclusive subsets.



$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$

Figure 2-15 Partitioning an event into several mutually exclusive subsets.

Total Probability Rule (two events)

For any events A and B,

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A') \qquad (2\text{-}7)$$

EXAMPLE 2-21

Consider the contamination discussion at the start of this section. Let F denote the event that the product fails, and let H denote the event that the chip is exposed to high levels of contamination. The requested probability is $P(F)$, and the information provided can be represented as

$$P(F|H) = 0.10 \quad \text{and} \quad P(F|H') = 0.005$$
$$P(H) = 0.20 \quad \text{and} \quad P(H') = 0.80$$

From Equation 2-7,

$$P(F) = 0.10(0.20) + 0.005(0.80) = 0.0235$$

which can be interpreted as just the weighted average of the two probabilities of failure.

The reasoning used to develop Equation 2-7 can be applied more generally. In the development of Equation 2-7, we only used the two mutually exclusive A and A'. However, the fact that $A \cup A' = S$, the entire sample space, was important. In general, a collection of sets $E_1, E_2, \ldots, E_k$ such that $E_1 \cup E_2 \cup \ldots \cup E_k = S$ is said to be **exhaustive.** A graphical display of partitioning an event B among a collection of mutually exclusive and exhaustive events is shown in Fig. 2-15 on page 43.

Total Probability Rule (multiple events)

Assume $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive sets. Then

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k) \end{aligned} \qquad (2\text{-}8)$$

EXAMPLE 2-22

Continuing with the semiconductor manufacturing example, assume the following probabilities for product failure subject to levels of contamination in manufacturing:

Probability of Failure	Level of Contamination
0.10	High
0.01	Medium
0.001	Low

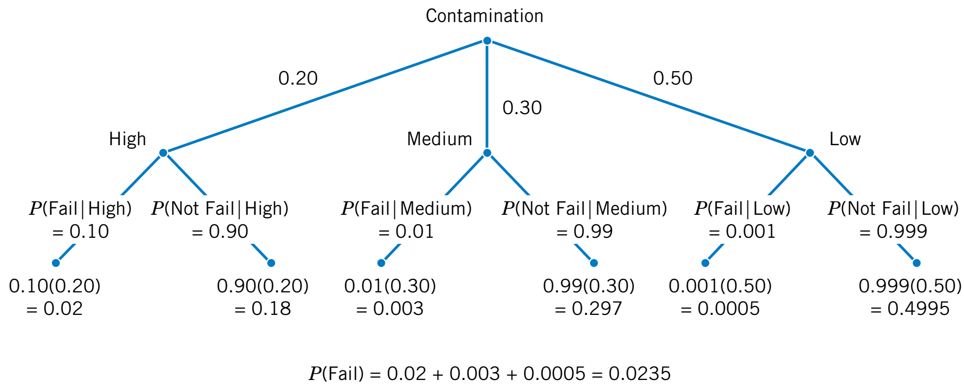


Figure 2-16 Tree diagram for Example 2-22.

In a particular production run, 20% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 50% to low levels of contamination. What is the probability that a product using one of these chips fails? Let

H denote the event that a chip is exposed to high levels of contamination

M denote the event that a chip is exposed to medium levels of contamination

L denote the event that a chip is exposed to low levels of contamination

Then,

$$P(F) = P(F|H)P(H) + P(F|M)P(M) + P(F|L)P(L)$$
$$= 0.10(0.20) + 0.01(0.30) + 0.001(0.50) = 0.0235$$

This problem is also conveniently solved using the tree diagram in Fig. 2-16.

EXERCISES FOR SECTION 2-5

2-70. Suppose that $P(A|B) = 0.4$ and $P(B) = 0.5$. Determine the following:
(a) $P(A \cap B)$
(b) $P(A' \cap B)$

2-71. Suppose that $P(A|B) = 0.2$, $P(A|B') = 0.3$, and $P(B) = 0.8$. What is $P(A)$?

2-72. The probability is 1% that an electrical connector that is kept dry fails during the warranty period of a portable computer. If the connector is ever wet, the probability of a failure during the warranty period is 5%. If 90% of the connectors are kept dry and 10% are wet, what proportion of connectors fail during the warranty period?

2-73. Suppose 2% of cotton fabric rolls and 3% of nylon fabric rolls contain flaws. Of the rolls used by a manufacturer, 70% are cotton and 30% are nylon. What is the probability that a randomly selected roll used by the manufacturer contains flaws?

2-74. In the manufacturing of a chemical adhesive, 3% of all batches have raw materials from two different lots. This occurs when holding tanks are replenished and the remaining portion of a lot is insufficient to fill the tanks.

Only 5% of batches with material from a single lot require reprocessing. However, the viscosity of batches consisting of two or more lots of material is more difficult to control, and 40% of such batches require additional processing to achieve the required viscosity.

Let A denote the event that a batch is formed from two different lots, and let B denote the event that a lot requires additional processing. Determine the following probabilities:
(a) $P(A)$ (b) $P(A')$
(c) $P(B|A)$ (d) $P(B|A')$
(e) $P(A \cap B)$ (f) $P(A \cap B')$
(g) $P(B)$

2-75. The edge roughness of slit paper products increases as knife blades wear. Only 1% of products slit with new blades have rough edges, 3% of products slit with blades of average sharpness exhibit roughness, and 5% of products slit with worn blades exhibit roughness. If 25% of the blades in manufacturing are new, 60% are of average sharpness, and 15% are worn, what is the proportion of products that exhibit edge roughness?

2-76. Samples of laboratory glass are in small, light packaging or heavy, large packaging. Suppose that 2 and 1% of the sample shipped in small and large packages, respectively, break during transit. If 60% of the samples are shipped in large packages and 40% are shipped in small packages, what proportion of samples break during shipment?

2-77. Incoming calls to a customer service center are classified as complaints (75% of call) or requests for information (25% of calls). Of the complaints, 40% deal with computer equipment that does not respond and 57% deal with incomplete software installation; and in the remaining 3% of complaints the user has improperly followed the installation instructions. The requests for information are evenly divided on technical questions (50%) and requests to purchase more products (50%).

(a) What is the probability that an incoming call to the customer service center will be from a customer who has not followed installation instructions properly?

(b) Find the probability that an incoming call is a request for purchasing more products.

2-78. Computer keyboard failures are due to faulty electrical connects (12%) or mechanical defects (88%). Mechanical defects are related to loose keys (27%) or improper assembly (73%). Electrical connect defects are caused by defective wires (35%), improper connections (13%), or poorly welded wires (52%).

(a) Find the probability that a failure is due to loose keys.

(b) Find the probability that a failure is due to improperly connected or poorly welded wires.

2-79. A batch of 25 injection-molded parts contains 5 that have suffered excessive shrinkage.

(a) If two parts are selected at random, and without replacement, what is the probability that the second part selected is one with excessive shrinkage?

(b) If three parts are selected at random, and without replacement, what is the probability that the third part selected is one with excessive shrinkage?

2-80. A lot of 100 semiconductor chips contains 20 that are defective.

(a) Two are selected, at random, without replacement, from the lot. Determine the probability that the second chip selected is defective.

(b) Three are selected, at random, without replacement, from the lot. Determine the probability that all are defective.

2-6 INDEPENDENCE

In some cases, the conditional probability of $P(B|A)$ might equal $P(B)$. In this special case, knowledge that the outcome of the experiment is in event A does not affect the probability that the outcome is in event B.

EXAMPLE 2-23

Suppose a day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Suppose two parts are selected from the batch, but the first part is replaced before the second part is selected. What is the probability that the second part is defective (denoted as B) given that the first part is defective (denoted as A)? The probability needed can be expressed as $P(B|A)$.

Because the first part is replaced prior to selecting the second part, the batch still contains 850 parts, of which 50 are defective. Therefore, the probability of B does not depend on whether or not the first part was defective. That is,

$$P(B|A) = 50/850$$

Also, the probability that both parts are defective is

$$P(A \cap B) = P(B|A)P(A) = \left(\frac{50}{850}\right) \cdot \left(\frac{50}{850}\right) = 0.0035$$

Table 2-4 Parts Classified

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	2	18	20
	No	38	342	380
	Total	40	360	400

EXAMPLE 2-24

The information in Table 2-3 related surface flaws to functionally defective parts. In that case, we determined that $P(D|F) = 10/40 = 0.25$ and $P(D) = 28/400 = 0.07$. Suppose that the situation is different and follows Table 2-4. Then,

$$P(D|F) = 2/40 = 0.05 \qquad \text{and} \qquad P(D) = 20/400 = 0.05$$

That is, the probability that the part is defective does not depend on whether it has surface flaws. Also,

$$P(F|D) = 2/20 = 0.10 \qquad \text{and} \qquad P(F) = 40/400 = 0.10$$

so the probability of a surface flaw does not depend on whether the part is defective. Furthermore, the definition of conditional probability implies that

$$P(F \cap D) = P(D|F)P(F)$$

but in the special case of this problem

$$P(F \cap D) = P(D)P(F) = \frac{2}{40} \cdot \frac{2}{20} = \frac{1}{200}$$

The preceding example illustrates the following conclusions. In the special case that $P(B \mid A) = P(B)$, we obtain

$$P(A \cap B) = P(B|A)P(A) = P(B)P(A)$$

and

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

These conclusions lead to an important definition.

Definition

Two events are **independent** if any one of the following equivalent statements is true:

(1) $P(A|B) = P(A)$

(2) $P(B|A) = P(B)$

(3) $P(A \cap B) = P(A)P(B)$ (2-9)

It is left as a mind-expanding exercise to show that independence implies related results such as

$$P(A' \cap B') = P(A')P(B').$$

The concept of independence is an important relationship between events and is used throughout this text. A mutually exclusive relationship between two events is based only on the outcomes that comprise the events. However, an independence relationship depends on the probability model used for the random experiment. Often, independence is assumed to be part of the random experiment that describes the physical system under study.

EXAMPLE 2-25

A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected at random, without replacement, from the batch. Let A denote the event that the first part is defective, and let B denote the event that the second part is defective.

We suspect that these two events are not independent because knowledge that the first part is defective suggests that it is less likely that the second part selected is defective. Indeed, $P(B|A) = 49/849$. Now, what is $P(B)$? Finding the unconditional $P(B)$ is somewhat difficult because the possible values of the first selection need to be considered:

$$\begin{aligned} P(B) &= P(B|A)P(A) + P(B|A')P(A') \\ &= (49/849)(50/850) + (50/849)(800/850) \\ &= 50/850 \end{aligned}$$

Interestingly, $P(B)$, the unconditional probability that the second part selected is defective, without any knowledge of the first part, is the same as the probability that the first part selected is defective. Yet, our goal is to assess independence. Because $P(B|A)$ does not equal $P(B)$, the two events are not independent, as we suspected.

When considering three or more events, we can extend the definition of independence with the following general result.

Definition

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots \times P(E_{i_k}) \qquad (2\text{-}10)$$

This definition is typically used to calculate the probability that several events occur assuming that they are independent and the individual event probabilities are known. The knowledge that the events are independent usually comes from a fundamental understanding of the random experiment.

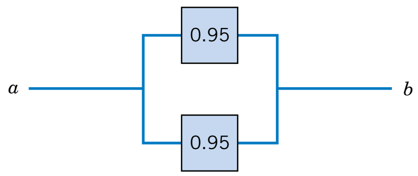
EXAMPLE 2-26

Assume that the probability that a wafer contains a large particle of contamination is 0.01 and that the wafers are independent; that is, the probability that a wafer contains a large particle is not dependent on the characteristics of any of the other wafers. If 15 wafers are analyzed, what is the probability that no large particles are found?

Let E_i denote the event that the ith wafer contains no large particles, $i = 1, 2, \ldots, 15$. Then, $P(E_i) = 0.99$. The probability requested can be represented as $P(E_1 \cap E_2 \cap \cdots \cap E_{15})$. From the independence assumption and Equation 2-10,

$$P(E_1 \cap E_2 \cap \cdots \cap E_{15}) = P(E_1) \times P(E_2) \times \cdots \times P(E_{15}) = 0.99^{15} = 0.86$$

EXAMPLE 2-27 The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



Let T and B denote the events that the top and bottom devices operate, respectively. There is a path if at least one device operates. The probability that the circuit operates is

$$P(T \text{ or } B) = 1 - P[(T \text{ or } B)'] = 1 - P(T' \text{ and } B')$$

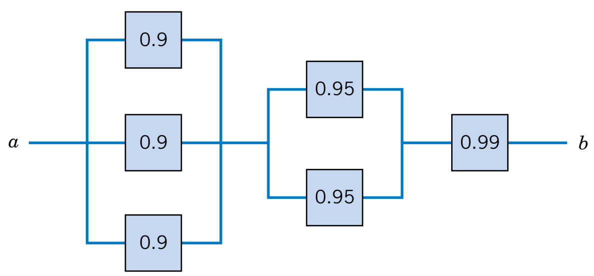
a simple formula for the solution can be derived from the complements T' and B'. From the independence assumption,

$$P(T' \text{ and } B') = P(T')P(B') = (1 - 0.95)^2 = 0.05^2$$

so

$$P(T \text{ or } B) = 1 - 0.05^2 = 0.9975$$

EXAMPLE 2-28 The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. What is the probability that the circuit operates?



The solution can be obtained from a partition of the graph into three columns. The probability that there is a path of functional devices only through the three units on the left can be determined from the independence in a manner similar to the previous example. It is

$$1 - 0.1^3$$

Similarly, the probability that there is a path of functional devices only through the two units in the middle is

$$1 - 0.05^2$$

The probability that there is a path of functional devices only through the one unit on the right is simply the probability that the device functions, namely, 0.99. Therefore, with the independence assumption used again, the solution is

$$(1 - 0.1^3)(1 - 0.05^2)(0.99) = 0.987$$

EXERCISES FOR SECTION 2-6

2-81. If $P(A|B) = 0.4$, $P(B) = 0.8$, and $P(A) = 0.5$, are the events A and B independent?

2-82. If $P(A|B) = 0.3$, $P(B) = 0.8$, and $P(A) = 0.3$, are the events B and the complement of A independent?

2-83. Disks of polycarbonate plastic from a supplier are analyzed for scratch and shock resistance. The results from 100 disks are summarized as follows:

		shock resistance	
		high	low
scratch resistance	high	70	9
	low	16	5

Let A denote the event that a disk has high shock resistance, and let B denote the event that a disk has high scratch resistance. Are events A and B independent?

2-84. Samples of a cast aluminum part are classified on the basis of surface finish (in microinches) and length measurements. The results of 100 parts are summarized as follows:

		length	
		excellent	good
surface finish	excellent	80	2
	good	10	8

Let A denote the event that a sample has excellent surface finish, and let B denote the event that a sample has excellent length. Are events A and B independent?

2-85. Samples of emissions from three suppliers are classified for conformance to air-quality specifications. The results from 100 samples are summarized as follows:

		conforms	
		yes	no
supplier	1	22	8
	2	25	5
	3	30	10

Let A denote the event that a sample is from supplier 1, and let B denote the event that a sample conforms to specifications.
(a) Are events A and B independent?
(b) Determine $P(B|A)$.

2-86. If $P(A) = 0.2$, $P(B) = 0.2$, and A and B are mutually exclusive, are they independent?

2-87. The probability that a lab specimen contains high levels of contamination is 0.10. Five samples are checked, and the samples are independent.
(a) What is the probability that none contains high levels of contamination?
(b) What is the probability that exactly one contains high levels of contamination?
(c) What is the probability that at least one contains high levels of contamination?

2-88. In a test of a printed circuit board using a random test pattern, an array of 10 bits is equally likely to be 0 or 1. Assume the bits are independent.
(a) What is the probability that all bits are 1s?
(b) What is the probability that all bits are 0s?
(c) What is the probability that exactly five bits are 1s and five bits are 0s?

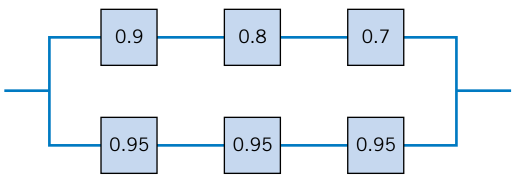
2-89. Eight cavities in an injection-molding tool produce plastic connectors that fall into a common stream. A sample is chosen every several minutes. Assume that the samples are independent.

(a) What is the probability that five successive samples were all produced in cavity one of the mold?

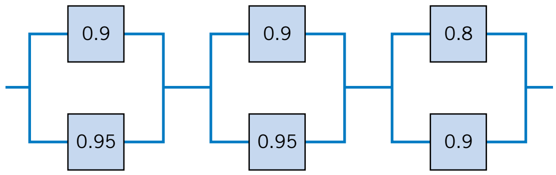
(b) What is the probability that five successive samples were all produced in the same cavity of the mold?

(c) What is the probability that four out of five successive samples were produced in cavity one of the mold?

2-90. The following circuit operates if and only if there is a path of functional devices from left to right. The probability that each device functions is as shown. Assume that the probability that a device is functional does not depend on whether or not other devices are functional. What is the probability that the circuit operates?



2-91. The following circuit operates if and only if there is a path of functional devices from left to right. The probability each device functions is as shown. Assume that the probability that a device functions does not depend on whether or not other devices are functional. What is the probability that the circuit operates?



2-92. An optical storage device uses an error recovery procedure that requires an immediate satisfactory readback of any written data. If the readback is not successful after three writing operations, that sector of the disk is eliminated as unacceptable for data storage. On an acceptable portion of the disk, the probability of a satisfactory readback is 0.98. Assume the readbacks are independent. What is the probability that an acceptable portion of the disk is eliminated as unacceptable for data storage?

2-93. A batch of 500 containers for frozen orange juice contains 5 that are defective. Two are selected, at random, without replacement, from the batch. Let A and B denote the events that the first and second container selected is defective, respectively.

(a) Are A and B independent events?

(b) If the sampling were done with replacement, would A and B be independent?

2-7 BAYES' THEOREM

In some examples, we do not have a complete table of information such as the parts in Table 2-3. We might know one conditional probability but would like to calculate a different one. In the semiconductor contamination problem in Example 2-22, we might ask the following: If the semiconductor chip in the product fails, what is the probability that the chip was exposed to high levels of contamination?

From the definition of conditional probability,

$$P(A \cap B) = P(A|B)P(B) = P(B \cap A) = P(B|A)P(A)$$

Now considering the second and last terms in the expression above, we can write

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \quad \text{for} \quad P(B) > 0 \qquad \text{(2-11)}$$

This is a useful result that enables us to solve for $P(A|B)$ in terms of $P(B|A)$.

EXAMPLE 2-29 We can answer the question posed at the start of this section as follows: The probability requested can be expressed as $P(H \mid F)$. Then,

$$P(H|F) = \frac{P(F|H)P(H)}{P(F)} = \frac{0.10(0.20)}{0.0235} = 0.85$$

The value of $P(F)$ in the denominator of our solution was found in Example 2-20.

In general, if $P(B)$ in the denominator of Equation 2-11 is written using the Total Probability Rule in Equation 2-8, we obtain the following general result, which is known as **Bayes' Theorem.**

Bayes' Theorem

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k)} \quad \text{(2-12)}$$

for $P(B) > 0$

EXAMPLE 2-30 Because a new medical procedure has been shown to be effective in the early detection of an illness, a medical screening of the population is proposed. The probability that the test correctly identifies someone with the illness as positive is 0.99, and the probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence of the illness in the general population is 0.0001. You take the test, and the result is positive. What is the probability that you have the illness?

Let D denote the event that you have the illness, and let S denote the event that the test signals positive. The probability requested can be denoted as $P(D|S)$. The probability that the test correctly signals someone without the illness as negative is 0.95. Consequently, the probability of a positive test without the illness is

$$P(S|D') = 0.05$$

From Bayes' Theorem,

$$\begin{aligned} P(D|S) &= P(S|D)P(D)/[P(S|D)P(D) + P(S|D')P(D')] \\ &= 0.99(0.0001)/[0.99(0.0001) + 0.05(1 - 0.0001)] \\ &= 1/506 = 0.002 \end{aligned}$$

Surprisingly, even though the test is effective, in the sense that $P(S|D)$ is high and $P(S|D')$ is low, because the incidence of the illness in the general population is low, the chances are quite small that you actually have the disease even if the test is positive.

EXERCISES FOR SECTION 2-7

2-94. Suppose that $P(A|B) = 0.7$, $P(A) = 0.5$, and $P(B) = 0.2$. Determine $P(B|A)$.

2-95. Software to detect fraud in consumer phone cards tracks the number of metropolitan areas where calls originate each day. It is found that 1% of the legitimate users originate calls from two or more metropolitan areas in a single day. However, 30% of fraudulent users originate calls from two or more metropolitan areas in a single day. The proportion of fraudulent users is 0.01%. If the same user originates calls from two or more metropolitan areas in a single day, what is the probability that the user is fraudulent?

2-96. Semiconductor lasers used in optical storage products require higher power levels for write operations than for read operations. High-power-level operations lower the useful life of the laser.

Lasers in products used for backup of higher speed magnetic disks primarily write, and the probability that the useful life exceeds five years is 0.95. Lasers that are in products that are used for main storage spend approximately an equal amount of time reading and writing, and the probability that the useful life exceeds five years is 0.995. Now, 25% of the products from a manufacturer are used for backup and 75% of the products are used for main storage.

Let A denote the event that a laser's useful life exceeds five years, and let B denote the event that a laser is in a product that is used for backup.

Use a tree diagram to determine the following:

(a) $P(B)$ (b) $P(A|B)$
(c) $P(A|B')$ (d) $P(A \cap B)$
(e) $P(A \cap B')$ (f) $P(A)$
(g) What is the probability that the useful life of a laser exceeds five years?
(h) What is the probability that a laser that failed before five years came from a product used for backup?

2-97. Customers are used to evaluate preliminary product designs. In the past, 95% of highly successful products received good reviews, 60% of moderately successful products received good reviews, and 10% of poor products received good reviews. In addition, 40% of products have been highly successful, 35% have been moderately successful, and 25% have been poor products.

(a) What is the probability that a product attains a good review?
(b) If a new design attains a good review, what is the probability that it will be a highly successful product?
(c) If a product does not attain a good review, what is the probability that it will be a highly successful product?

2-98. An inspector working for a manufacturing company has a 99% chance of correctly identifying defective items and a 0.5% chance of incorrectly classifying a good item as defective. The company has evidence that its line produces 0.9% of nonconforming items.

(a) What is the probability that an item selected for inspection is classified as defective?
(b) If an item selected at random is classified as nondefective, what is the probability that it is indeed good?

2-99. A new analytical method to detect pollutants in water is being tested. This new method of chemical analysis is important because, if adopted, it could be used to detect three different contaminants—organic pollutants, volatile solvents, and chlorinated compounds—instead of having to use a single test for each pollutant. The makers of the test claim that it can detect high levels of organic pollutants with 99.7% accuracy, volatile solvents with 99.95% accuracy, and chlorinated compounds with 89.7% accuracy. If a pollutant is not present, the test does not signal. Samples are prepared for the calibration of the test and 60% of them are contaminated with organic pollutants, 27% with volatile solvents, and 13% with traces of chlorinated compounds.

A test sample is selected randomly.

(a) What is the probability that the test will signal?
(b) If the test signals, what is the probability that chlorinated compounds are present?

2-8 RANDOM VARIABLES

We often summarize the outcome from a random experiment by a simple number. In many of the examples of random experiments that we have considered, the sample space has been a description of possible outcomes. In some cases, descriptions of outcomes are sufficient, but in other cases, it is useful to associate a number with each outcome in the sample space. Because the particular outcome of the experiment is not known in advance, the resulting value of our variable is not known in advance. For this reason, the variable that associates a number with the outcome of a random experiment is referred to as a **random variable.**

Definition

A **random variable** is a function that assigns a real number to each outcome in the sample space of a random experiment.

A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milliamperes.

Sometimes a measurement (such as current in a copper wire or length of a machined part) can assume any value in an interval of real numbers (at least theoretically). Then arbitrary precision in the measurement is possible. Of course, in practice, we might round off to the nearest tenth or hundredth of a unit. The random variable that represents this measurement is said to be a **continuous** random variable. The range of the random variable includes all values in an interval of real numbers; that is, the range can be thought of as a continuum.

In other experiments, we might record a count such as the number of transmitted bits that are received in error. Then the measurement is limited to integers. Or we might record that a proportion such as 0.0042 of the 10,000 transmitted bits were received in error. Then the measurement is fractional, but it is still limited to discrete points on the real line. Whenever the measurement is limited to discrete points on the real line, the random variable is said to be a **discrete** random variable.

Definition

A **discrete** random variable is a random variable with a finite (or countably infinite) range.
A **continuous** random variable is a random variable with an interval (either finite or infinite) of real numbers for its range.

In some cases, the random variable X is actually discrete but, because the range of possible values is so large, it might be more convenient to analyze X as a continuous random variable. For example, suppose that current measurements are read from a digital instrument that displays the current to the nearest one-hundredth of a milliampere. Because the possible measurements are limited, the random variable is discrete. However, it might be a more convenient, simple approximation to assume that the current measurements are values of a continuous random variable.

Examples of Random Variables

Examples of **continuous** random variables:
electrical current, length, pressure, temperature, time, voltage, weight

Examples of **discrete** random variables:
number of scratches on a surface, proportion of defective parts among 1000 tested, number of transmitted bits received in error.

EXERCISES FOR SECTION 2-8

2-100. Decide whether a discrete or continuous random variable is the best model for each of the following variables:
(a) The time until a projectile returns to earth.
(b) The number of times a transistor in a computer memory changes state in one operation.
(c) The volume of gasoline that is lost to evaporation during the filling of a gas tank.

(d) The outside diameter of a machined shaft.
(e) The number of cracks exceeding one-half inch in 10 miles of an interstate highway.
(f) The weight of an injection-molded plastic part.
(g) The number of molecules in a sample of gas.
(h) The concentration of output from a reactor.
(i) The current in an electronic circuit.

Supplemental Exercises

2-101. In circuit testing of printed circuit boards, each board either fails or does not fail the test. A board that fails the test is then checked further to determine which one of five defect types is the primary failure mode. Represent the sample space for this experiment.

2-102. The data from 200 machined parts are summarized as follows:

	depth of bore	
edge condition	above target	below target
coarse	15	10
moderate	25	20
smooth	50	80

(a) What is the probability that a part selected has a moderate edge condition and a below-target bore depth?
(b) What is the probability that a part selected has a moderate edge condition or a below-target bore depth?
(c) What is the probability that a part selected does not have a moderate edge condition or does not have a below-target bore depth?
(d) Construct a Venn diagram representation of the events in this sample space.

2-103. Computers in a shipment of 100 units contain a portable hard drive, CD RW drive, or both according to the following table:

	portable hard drive	
	yes	no
CD RW		
yes	15	80
no	4	1

Let A denote the events that a computer has a portable hard drive and let B denote the event that a computer has a CD RW drive. If one computer is selected randomly, compute
(a) $P(A)$ (b) $P(A \cap B)$
(c) $P(A \cup B)$ (d) $P(A' \cap B)$
(e) $P(A|B)$

2-104. The probability that a customer's order is not shipped on time is 0.05. A particular customer places three orders, and the orders are placed far enough apart in time that they can be considered to be independent events.
(a) What is the probability that all are shipped on time?
(b) What is the probability that exactly one is not shipped on time?
(c) What is the probability that two or more orders are not shipped on time?

2-105. Let E_1, E_2, and E_3 denote the samples that conform to a percentage of solids specification, a molecular weight specification, and a color specification, respectively. A total of 240 samples are classified by the E_1, E_2, and E_3 specifications, where *yes* indicates that the sample conforms.

E_3 yes

		E_2		
		yes	no	Total
E_1	yes	200	1	201
	no	5	4	9
Total		205	5	210

E_3 no

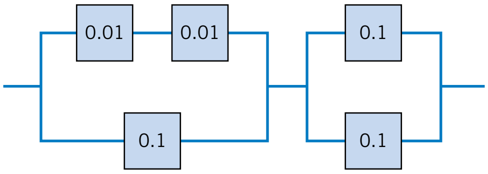
		E_2		
		yes	no	
E_1	yes	20	4	24
	no	6	0	6
Total		26	4	30

(a) Are E_1, E_2, and E_3 mutually exclusive events?
(b) Are E'_1, E'_2, and E'_3 mutually exclusive events?
(c) What is $P(E'_1 \text{ or } E'_2 \text{ or } E'_3)$?
(d) What is the probability that a sample conforms to all three specifications?
(e) What is the probability that a sample conforms to the E_1 or E_3 specification?
(f) What is the probability that a sample conforms to the E_1 or E_2 or E_3 specification?

2-106. Transactions to a computer database are either new items or changes to previous items. The addition of an item can be completed less than 100 milliseconds 90% of the time, but only 20% of changes to a previous item can be completed in less than this time. If 30% of transactions are changes, what is the probability that a transaction can be completed in less than 100 milliseconds?

2-107. A steel plate contains 20 bolts. Assume that 5 bolts are not torqued to the proper limit. Four bolts are selected at random, without replacement, to be checked for torque.
(a) What is the probability that all four of the selected bolts are torqued to the proper limit?
(b) What is the probability that at least one of the selected bolts is not torqued to the proper limit?

2-108. The following circuit operates if and only if there is a path of functional devices from left to right. Assume devices fail independently and that the probability of *failure* of each device is as shown. What is the probability that the circuit operates?



2-109. The probability of getting through by telephone to buy concert tickets is 0.92. For the same event, the probability of accessing the vendor's Web site is 0.95. Assume that these two ways to buy tickets are independent. What is the probability that someone who tries to buy tickets through the Internet and by phone will obtain tickets?

2-110. The British government has stepped up its information campaign regarding foot and mouth disease by mailing brochures to farmers around the country. It is estimated that 99% of Scottish farmers who receive the brochure possess enough information to deal with an outbreak of the disease, but only 90% of those without the brochure can deal with an outbreak. After the first three months of mailing, 95% of the farmers in Scotland received the informative brochure. Compute the probability that a randomly selected farmer will have enough information to deal effectively with an outbreak of the disease.

2-111. In an automated filling operation, the probability of an incorrect fill when the process is operated at a low speed is 0.001. When the process is operated at a high speed, the probability of an incorrect fill is 0.01. Assume that 30% of the containers are filled when the process is operated at a high speed and the remainder are filled when the process is operated at a low speed.

(a) What is the probability of an incorrectly filled container?
(b) If an incorrectly filled container is found, what is the probability that it was filled during the high-speed operation?

2-112. An encryption-decryption system consists of three elements: encode, transmit, and decode. A faulty encode occurs in 0.5% of the messages processed, transmission errors occur in 1% of the messages, and a decode error occurs in 0.1% of the messages. Assume the errors are independent.

(a) What is the probability of a completely defect-free message?
(b) What is the probability of a message that has either an encode or a decode error?

2-113. It is known that two defective copies of a commercial software program were erroneously sent to a shipping lot that has now a total of 75 copies of the program. A sample of copies will be selected from the lot without replacement.

(a) If three copies of the software are inspected, determine the probability that exactly one of the defective copies will be found.
(b) If three copies of the software are inspected, determine the probability that both defective copies will be found.
(c) If 73 copies are inspected, determine the probability that both copies will be found. Hint: Work with the copies that remain in the lot.

2-114. A robotic insertion tool contains 10 primary components. The probability that any component fails during the warranty period is 0.01. Assume that the components fail independently and that the tool fails if any component fails. What is the probability that the tool fails during the warranty period?

2-115. An e-mail message can travel through one of two server routes. The probability of transmission error in each of the servers and the proportion of messages that travel each route are shown in the following table. Assume that the servers are independent.

		probability of error			
	percentage of messages	server 1	server 2	server 3	server 4
route 1	30	0.01	0.015		
route 2	70			0.02	0.003

(a) What is the probability that a message will arrive without error?
(b) If a message arrives in error, what is the probability it was sent through route 1?

2-116. A machine tool is idle 15% of the time. You request immediate use of the tool on five different occasions during the year. Assume that your requests represent independent events.

(a) What is the probability that the tool is idle at the time of all of your requests?
(b) What is the probability that the machine is idle at the time of exactly four of your requests?
(c) What is the probability that the tool is idle at the time of at least three of your requests?

2-117. A lot of 50 spacing washers contains 30 washers that are thicker than the target dimension. Suppose that three washers are selected at random, without replacement, from the lot.

(a) What is the probability that all three washers are thicker than the target?
(b) What is the probability that the third washer selected is thicker than the target if the first two washers selected are thinner than the target?
(c) What is the probability that the third washer selected is thicker than the target?

2-118. Continuation of Exercise 2-117. Washers are selected from the lot at random, without replacement.

(a) What is the minimum number of washers that need to be selected so that the probability that all the washers are thinner than the target is less than 0.10?
(b) What is the minimum number of washers that need to be selected so that the probability that one or more washers are thicker than the target is at least 0.90?

2-119. The following table lists the history of 940 orders for features in an entry-level computer product.

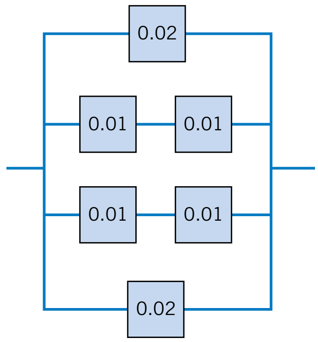
		extra memory	
		no	yes
optional high-	no	514	68
speed processor	yes	112	246

Let A be the event that an order requests the optional high-speed processor, and let B be the event that an order requests extra memory. Determine the following probabilities:

(a) $P(A \cup B)$ (b) $P(A \cap B)$
(c) $P(A' \cup B)$ (d) $P(A' \cap B')$
(e) What is the probability that an order requests an optional high-speed processor given that the order requests extra memory?
(f) What is the probability that an order requests extra memory given that the order requests an optional high-speed processor?

2-120. The alignment between the magnetic tape and head in a magnetic tape storage system affects the performance of the system. Suppose that 10% of the read operations are degraded by skewed alignments, 5% of the read operations are degraded by off-center alignments, and the remaining read operations are properly aligned. The probability of a read error is 0.01 from a skewed alignment, 0.02 from an off-center alignment, and 0.001 from a proper alignment.
(a) What is the probability of a read error?
(b) If a read error occurs, what is the probability that it is due to a skewed alignment?

2-121. The following circuit operates if and only if there is a path of functional devices from left to right. Assume that devices fail independently and that the probability of *failure* of each device is as shown. What is the probability that the circuit does not operate?



2-122. A company that tracks the use of its web site determined that the more pages a visitor views, the more likely the visitor is to provide contact information. Use the following tables to answer the questions:

Number of pages viewed:	1	2	3	4 or more
Percentage of visitors:	40	30	20	10
Percentage of visitors in each page-view catgory that provide contact information:	10	10	20	40

(a) What is the probability that a visitor to the web site provides contact information?
(b) If a visitor provides contact information, what is the probability that the visitor viewed four or more pages?

MIND-EXPANDING EXERCISES

2-123. The alignment between the magnetic tape and head in a magnetic tape storage system affects the performance of the system. Suppose that 10% of the read operations are degraded by skewed alignments, 5% by off-center alignments, 1% by both skewness and off-center, and the remaining read operations are properly aligned. The probability of a read error is 0.01 from a skewed alignment, 0.02 from an off-center alignment, 0.06 from both conditions, and 0.001 from a proper alignment. What is the probability of a read error.

2-124. Suppose that a lot of washers is large enough that it can be assumed that the sampling is done with replacement. Assume that 60% of the washers exceed the target thickness.

(a) What is the minimum number of washers that need to be selected so that the probability that all the washers are thinner than the target is less than 0.10?

(b) What is the minimum number of washers that need to be selected so that the probability that one or more washers are thicker than the target is at least 0.90?

2-125. A biotechnology manufacturing firm can produce diagnostic test kits at a cost of $20. Each kit for which there is a demand in the week of production can be sold for $100. However, the half-life of components in the kit requires the kit to be scrapped if it is not sold in the week of production. The cost of scrapping the kit is $5. The weekly demand is summarized as follows:

weekly demand				
Number of units	0	50	100	200
Probability of demand	0.05	0.4	0.3	0.25

How many kits should be produced each week to maximize the mean earnings of the firm?

2-126. Assume the following characteristics of the inspection process in Exercise 2-107. If an operator checks a bolt, the probability that an incorrectly torqued bolt is identified is 0.95. If a checked bolt is correctly torqued, the operator's conclusion is always correct. What is the probability that at least one bolt in the sample of four is identified as being incorrectly torqued?

2-127. If the events A and B are independent, show that A' and B' are independent.

2-128. Suppose that a table of part counts is generalized as follows:

		conforms	
		yes	no
supplier	1	ka	kb
	2	a	b

where a, b, and k are positive integers. Let A denote the event that a part is from supplier 1 and let B denote the event that a part conforms to specifications. Show that A and B are independent events.

This exercise illustrates the result that whenever the rows of a table (with r rows and c columns) are proportional, an event defined by a row category and an event defined by a column category are independent.

IMPORTANT TERMS AND CONCEPTS

In the E-book, click on any term or concept below to go to that subject.

Addition rule
Axioms of probability
Bayes' theorem
Conditional probability
Equally likely outcomes
Event
Independence
Multiplication rule
Mutually exclusive events
Outcome
Random experiment
Random variables discrete and continuous
Sample spaces—discrete and continuous
Total probability rule
With or without replacement

CD MATERIAL

Permutation
Combination

2-1.4 Counting Techniques (CD Only)

In many of the examples in Chapter 2, it is easy to determine the number of outcomes in each event. In more complicated examples, determining the number of outcomes that comprise the sample space (or an event) becomes more difficult. To associate probabilities with events, it is important to know the number of outcomes both in an event and in the sample space. Some simple rules can be used to simplify the calculations.

In Example 2-4, an automobile manufacturer provides vehicles equipped with selected options. Each vehicle is ordered

With or without an automatic transmission

With or without air conditioning

With one of three choices of a stereo system

With one of four exterior colors

The tree diagram in Fig. 2-6 describes the sample space of all possible vehicle types. The size of the sample space equals the number of branches in the last level of the tree and this quantity equals $2 \times 2 \times 3 \times 4 = 48$. This leads to the following useful result.

Multiplication Rule (for counting techniques)

If an operation can be described as a sequence of k steps, and

if the number of ways of completing step 1 is n_1, and

if the number of ways of completing step 2 is n_2 for each way of completing step 1, and

if the number of ways of completing step 3 is n_3 for each way of completing step 2, and so forth,

the total number of ways of completing the operation is

$$n_1 \times n_2 \times \cdots \times n_k$$

EXAMPLE S2-1 In the design of a casing for a gear housing, we can use four different types of fasteners, three different bolt lengths, and three different bolt locations. From the multiplication rule, $4 \times 3 \times 3 = 36$ different designs are possible.

Permutations

Another useful calculation is the number of ordered sequences of the elements of a set. Consider a set of elements, such as $S = \{a, b, c\}$. A **permutation** of the elements is an ordered sequence of the elements. For example, *abc*, *acb*, *bac*, *bca*, *cab*, and *cba* are all of the permutations of the elements of S.

The number of **permutations** of n different elements is $n!$ where

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \quad \text{(S2-1)}$$

This result follows from the multiplication rule. A permutation can be constructed by selecting the element to be placed in the first position of the sequence from the n elements, then selecting the element for the second position from the $n - 1$ remaining elements, then selecting the element for the third position from the remaining $n - 2$ elements, and so forth. Permutations such as these are sometimes referred to as linear permutations.

In some situations, we are interested in the number of arrangements of only some of the elements of a set. The following result also follows from the multiplication rule.

The number of permutations of a subset of r elements selected from a set of n different elements is

$$P_r^n = n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!} \qquad \text{(S2-2)}$$

EXAMPLE S2-2

A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many different designs are possible?

Each design consists of selecting a location from the eight locations for the first component, a location from the remaining seven for the second component, a location from the remaining six for the third component, and a location from the remaining five for the fourth component. Therefore,

$$P_4^8 = 8 \times 7 \times 6 \times 5 = \frac{8!}{4!} = 1680 \text{ different designs are possible.}$$

Sometimes we are interested in counting the number of ordered sequences for objects that are not all different. The following result is a useful, general calculation.

The number of permutations of $n = n_1 + n_2 + \cdots + n_r$ objects of which n_1 are of one type, n_2 are of a second type, . . . , and n_r are of an rth type is

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_r!} \qquad \text{(S2-3)}$$

EXAMPLE S2-3

Consider a machining operation in which a piece of sheet metal needs two identical diameter holes drilled and two identical size notches cut. We denote a drilling operation as d and a notching operation as n. In determining a schedule for a machine shop, we might be interested in the number of different possible sequences of the four operations. The number of possible sequences for two drilling operations and two notching operations is

$$\frac{4!}{2!\, 2!} = 6$$

The six sequences are easily summarized: *ddnn*, *dndn*, *dnnd*, *nddn*, *ndnd*, *nndd*.

EXAMPLE S2-4

A part is labeled by printing with four thick lines, three medium lines, and two thin lines. If each ordering of the nine lines represents a different label, how many different labels can be generated by using this scheme?

From Equation S2-3, the number of possible part labels is

$$\frac{9!}{4!\,3!\,2!} = 2520$$

Combinations

Another counting problem of interest is the number of subsets of r elements that can be selected from a set of n elements. Here, order is not important. Every subset of r elements can be indicated by listing the elements in the set and marking each element with a "*" if it is to be included in the subset. Therefore, each permutation of r *'s and $n - r$ blanks indicate a different subset and the number of these are obtained from Equation S2-3.

For example, if the set is $S = \{a, b, c, d\}$ the subset $\{a, c\}$ can be indicated as

$$\begin{matrix} a & b & c & d \\ * & & * & \end{matrix}$$

The number of subsets of size r that can be selected from a set of n elements is denoted as $\binom{n}{r}$ or C_r^n and

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{(S2-4)}$$

EXAMPLE S2-5

A printed circuit board has eight different locations in which a component can be placed. If five identical components are to be placed on the board, how many different designs are possible?

Each design is a subset of the eight locations that are to contain the components. From Equation S2-4, the number of possible designs is

$$\frac{8!}{5!\,3!} = 56$$

The following example uses the multiplication rule in combination with Equation S2-4 to answer a more difficult, but common, question.

EXAMPLE S2-6

A bin of 50 manufactured parts contains three defective parts and 47 nondefective parts. A sample of six parts is selected from the 50 parts. Selected parts are not replaced. That is, each part can only be selected once and the sample is a subset of the 50 parts. How many different samples are there of size six that contain exactly two defective parts?

A subset containing exactly two defective parts can be formed by first choosing the two defective parts from the three defective parts. Using Equation S2-4, this step can be completed in

$$\binom{3}{2} = \frac{3!}{2!\,1!} = 3 \text{ different ways}$$

Then, the second step is to select the remaining four parts from the 47 acceptable parts in the bin. The second step can be completed in

$$\binom{47}{4} = \frac{47!}{4!\,43!} = 178{,}365 \text{ different ways}$$

Therefore, from the multiplication rule, the number of subsets of size six that contain exactly two defective items is

$$3 \times 178{,}365 = 535{,}095$$

As an additional computation, the total number of different subsets of size six is found to be

$$\binom{50}{6} = \frac{50!}{6!\,44!} = 15{,}890{,}700$$

When probability is discussed in this chapter, the probability of an event is determined as the ratio of the number of outcomes in the event to the number of outcomes in the sample space (for equally likely outcomes). Therefore, the probability that a sample contains exactly two defective parts is

$$\frac{535{,}095}{15{,}890{,}700} = 0.034$$

Note that Example S2-7 illustrates the hypergeometric distribution.

EXERCISES FOR SECTION 2-1.4

S2-1. An order for a personal digital assistant can specify any one of five memory sizes, any one of three types of displays, any one of four sizes of a hard disk, and can either include or not include a pen tablet. How many different systems can be ordered?

S2-2. In a manufacturing operation, a part is produced by machining, polishing, and painting. If there are three machine tools, four polishing tools, and three painting tools, how many different routings (consisting of machining, followed by polishing, and followed by painting) for a part are possible?

S2-3. New designs for a wastewater treatment tank have proposed three possible shapes, four possible sizes, three locations for input valves, and four locations for output valves. How many different product designs are possible?

S2-4. A manufacturing process consists of 10 operations that can be completed in any order. How many different production sequences are possible?

S2-5. A manufacturing operations consists of 10 operations. However, five machining operations must be completed before any of the remaining five assembly operations can begin. Within each set of five, operations can be completed in any order. How many different production sequences are possible?

S2-6. In a sheet metal operation, three notches and four bends are required. If the operations can be done in any order, how many different ways of completing the manufacturing are possible?

S2-7. A lot of 140 semiconductor chips is inspected by choosing a sample of five chips. Assume 10 of the chips do not conform to customer requirements.

(a) How many different samples are possible?
(b) How many samples of five contain exactly one nonconforming chip?
(c) How many samples of five contain at least one nonconforming chip?

S2-8. In the layout of a printed circuit board for an electronic product, there are 12 different locations that can accommodate chips.

(a) If five different types of chips are to be placed on the board, how many different layouts are possible?

(b) If the five chips that are placed on the board are of the same type, how many different layouts are possible?

S2-9. In the laboratory analysis of samples from a chemical process, five samples from the process are analyzed daily. In addition, a control sample is analyzed two times each day to check the calibration of the laboratory instruments.

(a) How many different sequences of process and control samples are possible each day? Assume that the five process samples are considered identical and that the two control samples are considered identical.
(b) How many different sequences of process and control samples are possible if we consider the five process samples to be different and the two control samples to be identical.
(c) For the same situation as part (b), how many sequences are possible if the first test of each day must be a control sample?

S2-10. In the design of an electromechanical product, seven different components are to be stacked into a cylindrical casing that holds 12 components in a manner that minimizes the impact of shocks. One end of the casing is designated as the bottom and the other end is the top.

(a) How many different designs are possible?
(b) If the seven components are all identical, how many different designs are possible?
(c) If the seven components consist of three of one type of component and four of another type, how many different designs are possible? (more difficult)

S2-11. The design of a communication system considered the following questions:

(a) How many three-digit phone prefixes that are used to represent a particular geographic area (such as an area code) can be created from the digits 0 through 9?
(b) As in part (a), how many three-digit phone prefixes are possible that do not start with 0 or 1, but contain 0 or 1 as the middle digit?
(c) How many three-digit phone prefixes are possible in which no digit appears more than once in each prefix?

S2-12. A byte is a sequence of eight bits and each bit is either 0 or 1.

(a) How many different bytes are possible?
(b) If the first bit of a byte is a parity check, that is, the first byte is determined from the other seven bits, how many different bytes are possible?

S2-13. In a chemical plant, 24 holding tanks are used for final product storage. Four tanks are selected at random and without replacement. Suppose that six of the tanks contain material in which the viscosity exceeds the customer requirements.

(a) What is the probability that exactly one tank in the sample contains high viscosity material?
(b) What is the probability that at least one tank in the sample contains high viscosity material?
(c) In addition to the six tanks with high viscosity levels, four different tanks contain material with high impurities. What is the probability that exactly one tank in the sample contains high viscosity material and exactly one tank in the sample contains material with high impurities?

S2-14. Plastic parts produced by an injection-molding operation are checked for conformance to specifications. Each tool contains 12 cavities in which parts are produced, and these parts fall into a conveyor when the press opens. An inspector chooses 3 parts from among the 12 at random. Two cavities are affected by a temperature malfunction that results in parts that do not conform to specifications.

(a) What is the probability that the inspector finds exactly one nonconforming part?
(b) What is the probability that the inspector finds at least one nonconforming part?

S2-15. A bin of 50 parts contains five that are defective. A sample of two is selected at random, without replacement.

(a) Determine the probability that both parts in the sample are defective by computing a conditional probability.
(b) Determine the answer to part (a) by using the subset approach that was described in this section.