4 Continuous Random Variables and Probability Distributions

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Determine probabilities from probability density functions.
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability density functions, and the reverse.
3. Calculate means and variances for continuous random variables.
4. Understand the assumptions for each of the continuous probability distributions presented.
5. Select an appropriate continuous probability distribution to calculate probabilities in specific applications.
6. Calculate probabilities, determine means and variances for each of the continuous probability distributions presented.
7. Standardize normal random variables.

8. Use the table for the cumulative distribution function of a standard normal distribution to calculate probabilities.

9. Approximate probabilities for some binomial and Poisson distributions.

CD MATERIAL

10. Use continuity corrections to improve the normal approximation to those binomial and Poisson distributions.

Answers for most odd numbered exercises are at the end of the book. Answers to exercises whose numbers are surrounded by a box can be accessed in the e-Text by clicking on the box. Complete worked solutions to certain exercises are also available in the e-Text. These are indicated in the Answers to Selected Exercises section by a box around the exercise number. Exercises are also available for some of the text sections that appear on CD only. These exercises may be found within the e-Text immediately following the section they accompany.

4-1 CONTINUOUS RANDOM VARIABLES

Previously, we discussed the measurement of the current in a thin copper wire. We noted that the results might differ slightly in day-to-day replications because of small variations in variables that are not controlled in our experiment—changes in ambient temperatures, small impurities in the chemical composition of the wire, current source drifts, and so forth.

Another example is the selection of one part from a day's production and very accurately measuring a dimensional length. In practice, there can be small variations in the actual measured lengths due to many causes, such as vibrations, temperature fluctuations, operator differences, calibrations, cutting tool wear, bearing wear, and raw material changes. Even the measurement procedure can produce variations in the final results.

In these types of experiments, the measurement of interest—current in a copper wire experiment, length of a machined part—can be represented by a random variable. It is reasonable to model the range of possible values of the random variable by an interval (finite or infinite) of real numbers. For example, for the length of a machined part, our model enables the measurement from the experiment to result in any value within an interval of real numbers. Because the range is any value in an interval, the model provides for any precision in length measurements. However, because the number of possible values of the random variable X is uncountably infinite, X has a distinctly different distribution from the discrete random variables studied previously. The range of X includes all values in an interval of real numbers; that is, the range of X can be thought of as a continuum.

A number of continuous distributions frequently arise in applications. These distributions are described, and example computations of probabilities, means, and variances are provided in the remaining sections of this chapter.

4-2 PROBABILITY DISTRIBUTIONS AND PROBABILITY DENSITY FUNCTIONS

Density functions are commonly used in engineering to describe physical systems. For example, consider the density of a loading on a long, thin beam as shown in Fig. 4-1. For any point x along the beam, the density can be described by a function (in grams/cm). Intervals with large loadings correspond to large values for the function. The total loading between points a and b is determined as the integral of the density function from a to b. This integral is the area

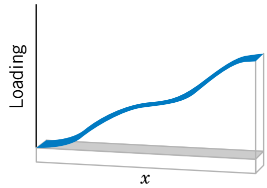


Figure 4-1 Density function of a loading on a long, thin beam.

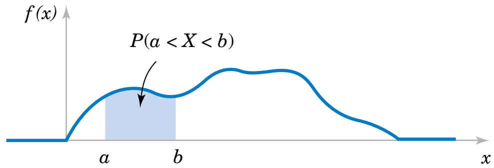


Figure 4-2 Probability determined from the area under $f(x)$.

under the density function over this interval, and it can be loosely interpreted as the sum of all the loadings over this interval.

Similarly, a **probability density function** $f(x)$ can be used to describe the probability distribution of a **continuous random variable** X. If an interval is likely to contain a value for X, its probability is large and it corresponds to large values for $f(x)$. The probability that X is between a and b is determined as the integral of $f(x)$ from a to b. See Fig. 4-2.

Definition

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

(3) $$P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx = \text{ area under } f(x) \text{ from } a \text{ to } b$$

for any a and b (4-1)

A probability density function provides a simple description of the probabilities associated with a random variable. As long as $f(x)$ is nonnegative and $\int_{-\infty}^{\infty} f(x)\, dx = 1$, $0 \leq P(a < X < b) \leq 1$ so that the probabilities are properly restricted. A probability density function is zero for x values that cannot occur and it is assumed to be zero wherever it is not specifically defined.

A **histogram** is an approximation to a probability density function. See Fig. 4-3. For each interval of the histogram, the area of the bar equals the relative frequency (proportion) of the measurements in the interval. The relative frequency is an estimate of the probability that a measurement falls in the interval. Similarly, the area under $f(x)$ over any interval equals the true probability that a measurement falls in the interval.

The important point is that **$f(x)$ is used to calculate an area** that represents the probability that X assumes a value in $[a, b]$. For the current measurement example, the probability that X results in [14 mA, 15 mA] is the integral of the probability density function of X over this interval. The probability that X results in [14.5 mA, 14.6 mA] is the integral of

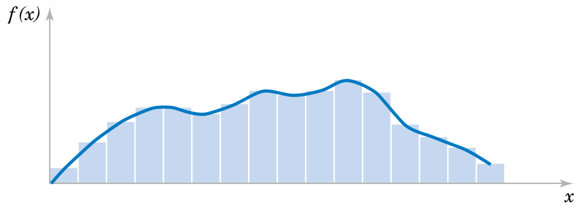


Figure 4-3 Histogram approximates a probability density function.

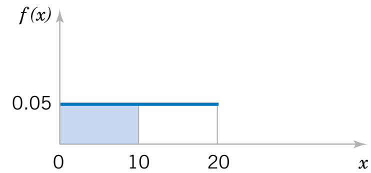


Figure 4-4 Probability density function for Example 4-1.

the same function, $f(x)$, over the smaller interval. By appropriate choice of the shape of $f(x)$, we can represent the probabilities associated with any continuous random variable X. The shape of $f(x)$ determines how the probability that X assumes a value in [14.5 mA, 14.6 mA] compares to the probability of any other interval of equal or different length.

For the density function of a loading on a long thin beam, because every point has zero width, the loading at any point is zero. Similarly, for a continuous random variable X and *any* value x.

$$P(X = x) = 0$$

Based on this result, it might appear that our model of a continuous random variable is useless. However, in practice, when a particular current measurement is observed, such as 14.47 milliamperes, this result can be interpreted as the rounded value of a current measurement that is actually in a range such as $14.465 \le x \le 14.475$. Therefore, the probability that the rounded value 14.47 is observed as the value for X is the probability that X assumes a value in the interval [14.465, 14.475], which is not zero. Similarly, because each point has zero probability, one need not distinguish between inequalities such as $<$ or $\le$ for continuous random variables.

If X is a **continuous random variable,** for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2) \quad \text{(4-2)}$$

EXAMPLE 4-1

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is $f(x) = 0.05$ for $0 \le x \le 20$. What is the probability that a current measurement is less than 10 milliamperes?

The probability density function is shown in Fig. 4-4. It is assumed that $f(x) = 0$ wherever it is not specifically defined. The probability requested is indicated by the shaded area in Fig. 4-4.

$$P(X < 10) = \int_0^{10} f(x)\, dx = \int_0^{10} 0.05\, dx = 0.5$$

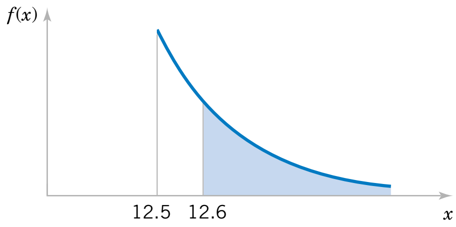


Figure 4-5 Probability density function for Example 4-2.

As another example,

$$P(5 < X < 20) = \int_{5}^{20} f(x)\,dx = 0.75$$

EXAMPLE 4-2

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \geq 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped? The density function and the requested probability are shown in Fig. 4-5. A part is scrapped if $X > 12.60$. Now,

$$P(X > 12.60) = \int_{12.6}^{\infty} f(x)\,dx = \int_{12.6}^{\infty} 20e^{-20(x-12.5)}\,dx = -e^{-20(x-12.5)}\Big|_{12.6}^{\infty} = 0.135$$

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x)\,dx = -e^{-20(x-12.5)}\Big|_{12.5}^{12.6} = 0.865$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865$.

EXERCISES FOR SECTION 4-2

4-1. Suppose that $f(x) = e^{-x}$ for $0 < x$. Determine the following probabilities:
(a) $P(1 < X)$ (b) $P(1 < X < 2.5)$
(c) $P(X = 3)$ (d) $P(X < 4)$
(e) $P(3 \leq X)$

4-2. Suppose that $f(x) = e^{-x}$ for $0 < x$.
(a) Determine x such that $P(x < X) = 0.10$.
(b) Determine x such that $P(X \leq x) = 0.10$.

4-3. Suppose that $f(x) = x/8$ for $3 < x < 5$. Determine the following probabilities:
(a) $P(X < 4)$ (b) $P(X > 3.5)$
(c) $P(4 < X < 5)$ (d) $P(X < 4.5)$
(e) $P(X < 3.5$ or $X > 4.5)$

4-4. Suppose that $f(x) = e^{-(x-4)}$ for $4 < x$. Determine the following probabilities:
(a) $P(1 < X)$ (b) $P(2 \leq X < 5)$

(c) $P(5 < X)$ (d) $P(8 < X < 12)$
(e) Determine x such that $P(X < x) = 0.90$.

4-5. Suppose that $f(x) = 1.5x^2$ for $-1 < x < 1$. Determine the following probabilities:
(a) $P(0 < X)$ (b) $P(0.5 < X)$
(c) $P(-0.5 \le X \le 0.5)$ (d) $P(X < -2)$
(e) $P(X < 0 \text{ or } X > -0.5)$
(f) Determine x such that $P(x < X) = 0.05$.

4-6. The probability density function of the time to failure of an electronic component in a copier (in hours) is $f(x) = \frac{e^{-x/1000}}{1000}$ for $x > 0$. Determine the probability that
(a) A component lasts more than 3000 hours before failure.
(b) A component fails in the interval from 1000 to 2000 hours.
(c) A component fails before 1000 hours.
(d) Determine the number of hours at which 10% of all components have failed.

4-7. The probability density function of the net weight in pounds of a packaged chemical herbicide is $f(x) = 2.0$ for $49.75 < x < 50.25$ pounds.
(a) Determine the probability that a package weighs more than 50 pounds.
(b) How much chemical is contained in 90% of all packages?

4-8. The probability density function of the length of a hinge for fastening a door is $f(x) = 1.25$ for $74.6 < x < 75.4$ millimeters. Determine the following:
(a) $P(X < 74.8)$
(b) $P(X < 74.8 \quad \text{or} \quad X > 75.2)$
(c) If the specifications for this process are from 74.7 to 75.3 millimeters, what proportion of hinges meets specifications?

4-9. The probability density function of the length of a metal rod is $f(x) = 2$ for $2.3 < x < 2.8$ meters.
(a) If the specifications for this process are from 2.25 to 2.75 meters, what proportion of the bars fail to meet the specifications?
(b) Assume that the probability density function is $f(x) = 2$ for an interval of length 0.5 meters. Over what value should the density be centered to achieve the greatest proportion of bars within specifications?

4-10. If X is a continuous random variable, argue that $P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2)$.

4-3 CUMULATIVE DISTRIBUTION FUNCTIONS

An alternative method to describe the distribution of a discrete random variable can also be used for continuous random variables.

Definition

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du \qquad (4\text{-}3)$$

for $-\infty < x < \infty$.

Extending the definition of $f(x)$ to the entire real line enables us to define the cumulative distribution function for all real numbers. The following example illustrates the definition.

EXAMPLE 4-3

For the copper current measurement in Example 4-1, the cumulative distribution function of the random variable X consists of three expressions. If $x < 0$, $f(x) = 0$. Therefore,

$$F(x) = 0, \quad \text{for} \quad x < 0$$

and

$$F(x) = \int_0^x f(u)\,du = 0.05x, \quad \text{for} \quad 0 \le x < 20$$

Finally,

$$F(x) = \int_0^x f(u)\,du = 1, \quad \text{for} \quad 20 \le x$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.05x & 0 \le x < 20 \\ 1 & 20 \le x \end{cases}$$

The plot of $F(x)$ is shown in Fig. 4-6.

Notice that in the definition of $F(x)$ any $<$ can be changed to $\le$ and vice versa. That is, $F(x)$ can be defined as either $0.05x$ or 0 at the end-point $x = 0$, and $F(x)$ can be defined as either $0.05x$ or 1 at the end-point $x = 20$. In other words, $F(x)$ is a continuous function. For a discrete random variable, $F(x)$ is not a continuous function. Sometimes, a continuous random variable is defined as one that has a continuous cumulative distribution function.

EXAMPLE 4-4 For the drilling operation in Example 4-2, $F(x)$ consists of two expressions.

$$F(x) = 0 \quad \text{for} \quad x < 12.5$$

and for $12.5 \le x$

$$F(x) = \int_{12.5}^{x} 20e^{-20(u-12.5)}\,du = 1 - e^{-20(x-12.5)}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & 12.5 \le x \end{cases}$$

Figure 4-7 displays a graph of $F(x)$.

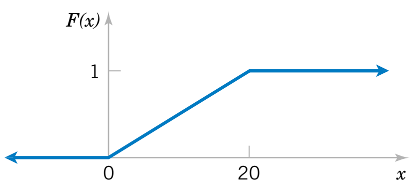


Figure 4-6 Cumulative distribution function for Example 4-3.

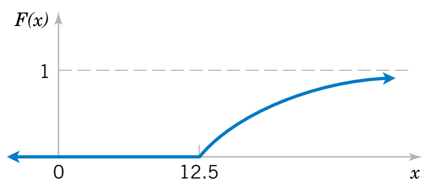


Figure 4-7 Cumulative distribution function for Example 4-4.

The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating. Recall that the fundamental theorem of calculus states that

$$\frac{d}{dx}\int_{-\infty}^{x} f(u)\,du = f(x)$$

Then, given $F(x)$

$$f(x) = \frac{dF(x)}{dx}$$

as long as the derivative exists.

EXAMPLE 4-5

The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.01x} & 0 \le x \end{cases}$$

Determine the probability density function of X. What proportion of reactions is complete within 200 milliseconds? Using the result that the probability density function is the derivative of the $F(x)$, we obtain

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.01e^{-0.01x} & 0 \le x \end{cases}$$

The probability that a reaction completes within 200 milliseconds is

$$P(X < 200) = F(200) = 1 - e^{-2} = 0.8647.$$

EXERCISES FOR SECTION 4-3

4-11. Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2x & 0 \le x < 5 \\ 1 & 5 \le x \end{cases}$$

Determine the following:
(a) $P(X < 2.8)$ (b) $P(X > 1.5)$
(c) $P(X < -2)$ (d) $P(X > 6)$

4-12. Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.25x + 0.5 & -2 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

Determine the following:
(a) $P(X < 1.8)$ (b) $P(X > -1.5)$
(c) $P(X < -2)$ (d) $P(-1 < X < 1)$

4-13. Determine the cumulative distribution function for the distribution in Exercise 4-1.

4-14. Determine the cumulative distribution function for the distribution in Exercise 4-3.

4-15. Determine the cumulative distribution function for the distribution in Exercise 4-4.

4-16. Determine the cumulative distribution function for the distribution in Exercise 4-6. Use the cumulative distribution function to determine the probability that a component lasts more than 3000 hours before failure.

4-17. Determine the cumulative distribution function for the distribution in Exercise 4-8. Use the cumulative distribution function to determine the probability that a length exceeds 75 millimeters.

Determine the probability density function for each of the following cumulative distribution functions.

4-18. $F(x) = 1 - e^{-2x} \quad x > 0$

4-19.

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2x & 0 \le x < 4 \\ 0.04x + 0.64 & 4 \le x < 9 \\ 1 & 9 \le x \end{cases}$$

4-20.

$$F(x) = \begin{cases} 0 & x < -2 \\ 0.25x + 0.5 & -2 \le x < 1 \\ 0.5x + 0.25 & 1 \le x < 1.5 \\ 1 & 1.5 \le x \end{cases}$$

4-21. The gap width is an important property of a magnetic recording head. In coded units, if the width is a continuous random variable over the range from $0 < x < 2$ with $f(x) = 0.5x$, determine the cumulative distribution function of the gap width.

4-4 MEAN AND VARIANCE OF A CONTINUOUS RANDOM VARIABLE

The mean and variance of a continuous random variable are defined similarly to a discrete random variable. Integration replaces summation in the definitions. If a probability density function is viewed as a loading on a beam as in Fig. 4-1, the mean is the balance point.

Definition

Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \tag{4-4}$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

The equivalence of the two formulas for variance can be derived as one, as was done for discrete random variables.

EXAMPLE 4-6

For the copper current measurement in Example 4-1, the mean of X is

$$E(X) = \int_{0}^{20} x f(x)\, dx = 0.05x^2/2 \Big|_{0}^{20} = 10$$

The variance of X is

$$V(X) = \int_{0}^{20} (x - 10)^2 f(x)\, dx = 0.05(x - 10)^3/3 \Big|_{0}^{20} = 33.33$$

The expected value of a function $h(X)$ of a continuous random variable is defined similarly to a function of a discrete random variable.

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)\,dx \tag{4-5}$$

EXAMPLE 4-7

In Example 4-1, X is the current measured in milliamperes. What is the expected value of the squared current? Now, $h(X) = X^2$. Therefore,

$$E[h(X)] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{0}^{20} 0.05x^2\,dx = 0.05\frac{x^3}{3}\Bigg|_0^{20} = 133.33$$

In the previous example, the expected value of X^2 does not equal $E(X)$ squared. However, in the special case that $h(X) = aX + b$ for any constants a and b, $E[h(X)] = aE(X) + b$. This can be shown from the properties of integrals.

EXAMPLE 4-8

For the drilling operation in Example 4-2, the mean of X is

$$E(X) = \int_{12.5}^{\infty} xf(x)\,dx = \int_{12.5}^{\infty} x\,20e^{-20(x-12.5)}\,dx$$

Integration by parts can be used to show that

$$E(X) = -xe^{-20(x-12.5)} - \frac{e^{-20(x-12.5)}}{20}\Bigg|_{12.5}^{\infty} = 12.5 + 0.05 = 12.55$$

The variance of X is

$$V(X) = \int_{12.5}^{\infty} (x - 12.55)^2 f(x)\,dx$$

Although more difficult, integration by parts can be used two times to show that $V(X) = 0.0025$.

EXERCISES FOR SECTION 4-4

4-22. Suppose $f(x) = 0.25$ for $0 < x < 4$. Determine the mean and variance of X.

4-23. Suppose $f(x) = 0.125x$ for $0 < x < 4$. Determine the mean and variance of X.

4-24. Suppose $f(x) = 1.5x^2$ for $-1 < x < 1$. Determine the mean and variance of X.

4-25. Suppose that $f(x) = x/8$ for $3 < x < 5$. Determine the mean and variance for x.

4-26. Determine the mean and variance of the weight of packages in Exercise 4.7.

4-27. The thickness of a conductive coating in micrometers has a density function of $600x^{-2}$ for $100\ \mu\text{m} < x < 120\ \mu\text{m}$.

(a) Determine the mean and variance of the coating thickness.
(b) If the coating costs \$0.50 per micrometer of thickness on each part, what is the average cost of the coating per part?

4-28. Suppose that contamination particle size (in micrometers) can be modeled as $f(x) = 2x^{-3}$ for $1 < x$. Determine the mean of X.

4-29. Integration by parts is required. The probability density function for the diameter of a drilled hole in millimeters is $10e^{-10(x-5)}$ for $x > 5$ mm. Although the target diameter is 5 millimeters, vibrations, tool wear, and other nuisances produce diameters larger than 5 millimeters.

(a) Determine the mean and variance of the diameter of the holes.
(b) Determine the probability that a diameter exceeds 5.1 millimeters.

4-30. Suppose the probability density function of the length of computer cables is $f(x) = 0.1$ from 1200 to 1210 millimeters.

(a) Determine the mean and standard deviation of the cable length.
(b) If the length specifications are $1195 < x < 1205$ millimeters, what proportion of cables are within specifications?

4-5 CONTINUOUS UNIFORM DISTRIBUTION

The simplest continuous distribution is analogous to its discrete counterpart.

Definition

A continuous random variable X with probability density function

$$f(x) = 1/(b - a), \qquad a \le x \le b \tag{4-6}$$

is a **continuous uniform random variable.**

The probability density function of a continuous uniform random variable is shown in Fig. 4-8. The mean of the continuous uniform random variable X is

$$E(X) = \int_a^b \frac{x}{b-a}\,dx = \left.\frac{0.5x^2}{b-a}\right|_a^b = \frac{(a+b)}{2}$$

The variance of X is

$$V(X) = \int_a^b \frac{\left(x - \left(\frac{a+b}{2}\right)\right)^2}{b-a}\,dx = \left.\frac{\left(x - \frac{a+b}{2}\right)^3}{3(b-a)}\right|_a^b = \frac{(b-a)^2}{12}$$

These results are summarized as follows.

If X is a continuous uniform random variable over $a \le x \le b$,

$$\mu = E(X) = \frac{(a+b)}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12} \tag{4-7}$$

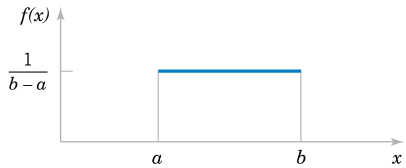


Figure 4-8 Continuous uniform probability density function.

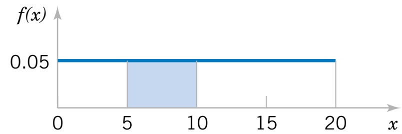


Figure 4-9 Probability for Example 4-9.

EXAMPLE 4-9 Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is $f(x) = 0.05, 0 \le x \le 20$.

What is the probability that a measurement of current is between 5 and 10 milliamperes? The requested probability is shown as the shaded area in Fig. 4-9.

$$P(5 < X < 10) = \int_5^{10} f(x)\,dx$$
$$= 5(0.05) = 0.25$$

The mean and variance formulas can be applied with $a = 0$ and $b = 20$. Therefore,

$$E(X) = 10 \text{ mA} \quad \text{and} \quad V(X) = 20^2/12 = 33.33 \text{ mA}^2$$

Consequently, the standard deviation of X is 5.77 mA.

The cumulative distribution function of a continuous uniform random variable is obtained by integration. If $a < x < b$,

$$F(x) = \int_a^x 1/(b-a)\,du = x/(b-a) - a/(b-a)$$

Therefore, the complete description of the cumulative distribution function of a continuous uniform random variable is

$$F(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

An example of $F(x)$ for a continuous uniform random variable is shown in Fig. 4-6.

EXERCISES FOR SECTION 4-5

4-31. Suppose X has a continuous uniform distribution over the interval [1.5, 5.5].
(a) Determine the mean, variance, and standard deviation of X.
(b) What is $P(X < 2.5)$?

4-32. Suppose X has a continuous uniform distribution over the interval $[-1, 1]$.
(a) Determine the mean, variance, and standard deviation of X.
(b) Determine the value for x such that $P(-x < X < x) = 0.90$.

4-33. The net weight in pounds of a packaged chemical herbicide is uniform for $49.75 < x < 50.25$ pounds.
(a) Determine the mean and variance of the weight of packages.

(b) Determine the cumulative distribution function of the weight of packages.
(c) Determine $P(X < 50.1)$.

4-34. The thickness of a flange on an aircraft component is uniformly distributed between 0.95 and 1.05 millimeters.
(a) Determine the cumulative distribution function of flange thickness.
(b) Determine the proportion of flanges that exceeds 1.02 millimeters.
(c) What thickness is exceeded by 90% of the flanges?
(d) Determine the mean and variance of flange thickness.

4-35. Suppose the time it takes a data collection operator to fill out an electronic form for a database is uniformly between 1.5 and 2.2 minutes.
(a) What is the mean and variance of the time it takes an operator to fill out the form?
(b) What is the probability that it will take less than two minutes to fill out the form?
(c) Determine the cumulative distribution function of the time it takes to fill out the form.

4-36. The probability density function of the time it takes a hematology cell counter to complete a test on a blood sample is $f(x) = 0.2$ for $50 < x < 75$ seconds.
(a) What percentage of tests require more than 70 seconds to complete.
(b) What percentage of tests require less than one minute to complete.
(c) Determine the mean and variance of the time to complete a test on a sample.

4-37. The thickness of photoresist applied to wafers in semiconductor manufacturing at a particular location on the wafer is uniformly distributed between 0.2050 and 0.2150 micrometers.
(a) Determine the cumulative distribution function of photoresist thickness.
(b) Determine the proportion of wafers that exceeds 0.2125 micrometers in photoresist thickness.
(c) What thickness is exceeded by 10% of the wafers?
(d) Determine the mean and variance of photoresist thickness.

4-38. The probability density function of the time required to complete an assembly operation is $f(x) = 0.1$ for $30 < x < 40$ seconds.
(a) Determine the proportion of assemblies that requires more than 35 seconds to complete.
(b) What time is exceeded by 90% of the assemblies?
(c) Determine the mean and variance of time of assembly.

4-6 NORMAL DISTRIBUTION

Undoubtedly, the most widely used model for the distribution of a random variable is a **normal distribution.** Whenever a random experiment is replicated, the random variable that equals the average (or total) result over the replicates tends to have a normal distribution as the number of replicates becomes large. De Moivre presented this fundamental result, known as the **central limit theorem,** in 1733. Unfortunately, his work was lost for some time, and Gauss independently developed a normal distribution nearly 100 years later. Although De Moivre was later credited with the derivation, a normal distribution is also referred to as a **Gaussian** distribution.

When do we average (or total) results? Almost always. For example, an automotive engineer may plan a study to average pull-off force measurements from several connectors. If we assume that each measurement results from a replicate of a random experiment, the normal distribution can be used to make approximate conclusions about this average. These conclusions are the primary topics in the subsequent chapters of this book.

Furthermore, sometimes the central limit theorem is less obvious. For example, assume that the deviation (or error) in the length of a machined part is the sum of a large number of infinitesimal effects, such as temperature and humidity drifts, vibrations, cutting angle variations, cutting tool wear, bearing wear, rotational speed variations, mounting and fixturing variations, variations in numerous raw material characteristics, and variation in levels of contamination. If the component errors are independent and equally likely to be positive or negative, the total error can be shown to have an approximate normal distribution. Furthermore, the normal distribution arises in the study of numerous basic physical phenomena. For example, the physicist Maxwell developed a normal distribution from simple assumptions regarding the velocities of molecules.

The theoretical basis of a normal distribution is mentioned to justify the somewhat complex form of the probability density function. Our objective now is to calculate probabilities for a normal random variable. The central limit theorem will be stated more carefully later.

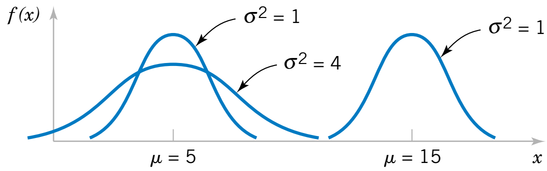


Figure 4-10 Normal probability density functions for selected values of the parameters μ and σ^2.

Random variables with different means and variances can be modeled by normal probability density functions with appropriate choices of the center and width of the curve. The value of $E(X) = \mu$ determines the center of the probability density function and the value of $V(X) = \sigma^2$ determines the width. Figure 4-10 illustrates several normal probability density functions with selected values of μ and σ^2. Each has the characteristic symmetric bell-shaped curve, but the centers and dispersions differ. The following definition provides the formula for normal probability density functions.

Definition

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty \tag{4-8}$$

is a **normal random variable** with parameters μ, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2 \tag{4-9}$$

and the notation $N(\mu, \sigma^2)$ is used to denote the distribution. The mean and variance of X are shown to equal μ and σ^2, respectively, at the end of this Section 5-6.

EXAMPLE 4-10 Assume that the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 $(\text{milliamperes})^2$. What is the probability that a measurement exceeds 13 milliamperes?

Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. This probability is shown as the shaded area under the normal probability density function in Fig. 4-11. Unfortunately, there is no closed-form expression for the integral of a normal probability density function, and probabilities based on the normal distribution are typically found numerically or from a table (that we will later introduce).

Some useful results concerning a normal distribution are summarized below and in Fig. 4-12. For any normal random variable,

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

Also, from the symmetry of $f(x)$, $P(X > \mu) = P(X < \mu) = 0.5$. Because $f(x)$ is positive for all x, this model assigns some probability to each interval of the real line. However, the

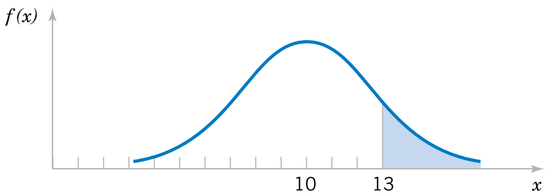


Figure 4-11 Probability that $X > 13$ for a normal random variable with $\mu = 10$ and $\sigma^2 = 4$.

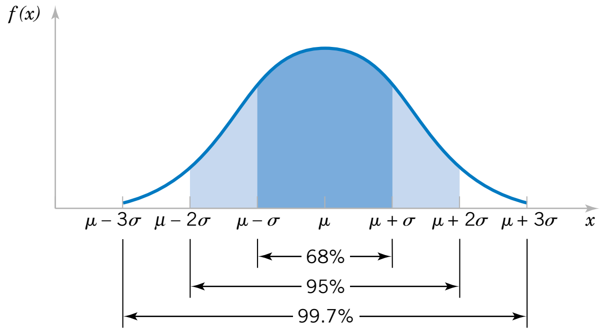


Figure 4-12 Probabilities associated with a normal distribution.

probability density function decreases as x moves farther from μ. Consequently, the probability that a measurement falls far from μ is small, and at some distance from μ the probability of an interval can be approximated as zero.

The area under a normal probability density function beyond 3σ from the mean is quite small. This fact is convenient for quick, rough sketches of a normal probability density function. The sketches help us determine probabilities. Because more than 0.9973 of the probability of a normal distribution is within the interval $(\mu - 3\sigma, \mu + 3\sigma)$, 6σ is often referred to as the **width** of a normal distribution. Advanced integration methods can be used to show that the area under the normal probability density function from $-\infty < x < \infty$ is 1.

Definition

A normal random variable with

$$\mu = 0 \quad \text{and} \quad \sigma^2 = 1$$

is called a **standard normal random variable** and is denoted as Z.

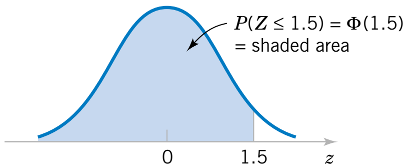
The cumulative distribution function of a standard normal random variable is denoted as

$$\Phi(z) = P(Z \leq z)$$

Appendix Table II provides cumulative probability values for $\Phi(z)$, for a standard normal random variable. Cumulative distribution functions for normal random variables are also widely available in computer packages. They can be used in the same manner as Appendix Table II to obtain probabilities for these random variables. The use of Table II is illustrated by the following example.

EXAMPLE 4-11 Assume Z is a standard normal random variable. Appendix Table II provides probabilities of the form $P(Z \leq z)$. The use of Table II to find $P(Z \leq 1.5)$ is illustrated in Fig. 4-13. Read down the z column to the row that equals 1.5. The probability is read from the adjacent column, labeled 0.00, to be 0.93319.

The column headings refer to the hundredth's digit of the value of z in $P(Z \leq z)$. For example, $P(Z \leq 1.53)$ is found by reading down the z column to the row 1.5 and then selecting the probability from the column labeled 0.03 to be 0.93699.



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 4-13 Standard normal probability density function.

Probabilities that are not of the form $P(Z \le z)$ are found by using the basic rules of probability and the symmetry of the normal distribution along with Appendix Table II. The following examples illustrate the method.

EXAMPLE 4-12 The following calculations are shown pictorially in Fig. 4-14. In practice, a probability is often rounded to one or two significant digits.

(1) $P(Z > 1.26) = 1 - P(Z \le 1.26) = 1 - 0.89616 = 0.10384$

(2) $P(Z < -0.86) = 0.19490.$

(3) $P(Z > -1.37) = P(Z < 1.37) = 0.91465$

(4) $P(-1.25 < Z < 0.37)$. This probability can be found from the difference of two areas, $P(Z < 0.37) - P(Z < -1.25)$. Now,

$$P(Z < 0.37) = 0.64431 \quad \text{and} \quad P(Z < -1.25) = 0.10565$$

Therefore,

$$P(-1.25 < Z < 0.37) = 0.64431 - 0.10565 = 0.53866$$

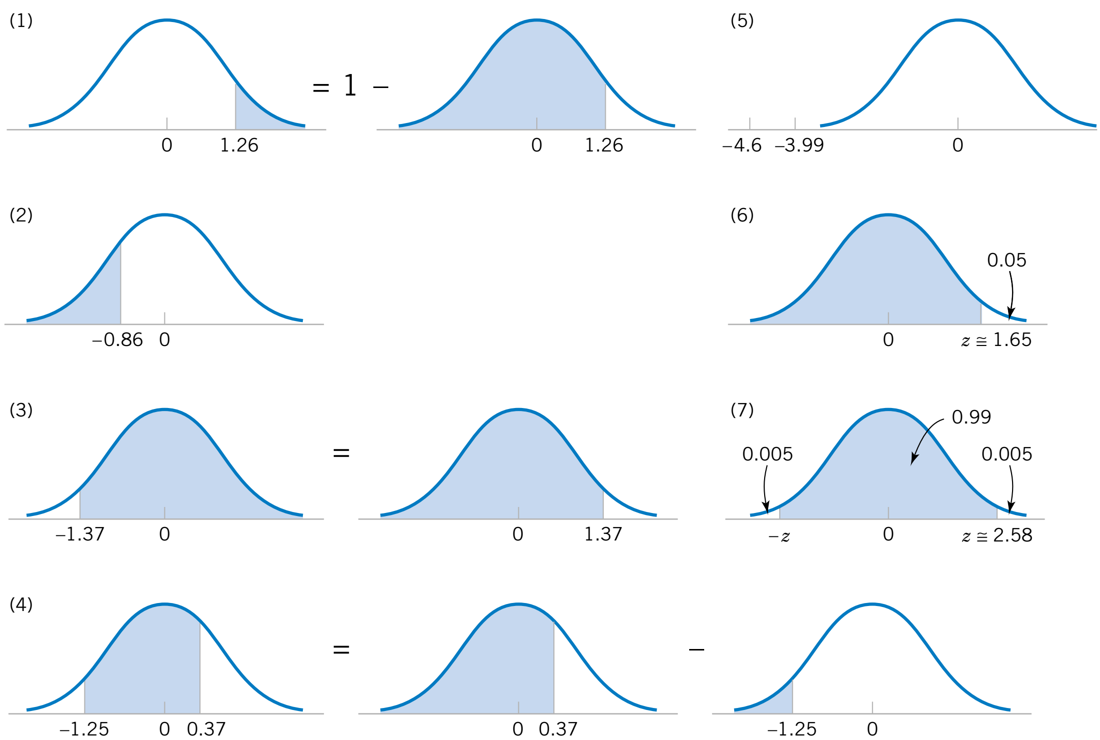


Figure 4-14 Graphical displays for standard normal distributions.

(5) $P(Z \leq -4.6)$ cannot be found exactly from Appendix Table II. However, the last entry in the table can be used to find that $P(Z \leq -3.99) = 0.00003$. Because $P(Z \leq -4.6) < P(Z \leq -3.99)$, $P(Z \leq -4.6)$ is nearly zero.

(6) Find the value z such that $P(Z > z) = 0.05$. This probability expression can be written as $P(Z \leq z) = 0.95$. Now, Table II is used in reverse. We search through the probabilities to find the value that corresponds to 0.95. The solution is illustrated in Fig. 4-14. We do not find 0.95 exactly; the nearest value is 0.95053, corresponding to $z = 1.65$.

(7) Find the value of z such that $P(-z < Z < z) = 0.99$. Because of the symmetry of the normal distribution, if the area of the shaded region in Fig. 4-14(7) is to equal 0.99, the area in each tail of the distribution must equal 0.005. Therefore, the value for z corresponds to a probability of 0.995 in Table II. The nearest probability in Table II is 0.99506, when $z = 2.58$.

The preceding examples show how to calculate probabilities for standard normal random variables. To use the same approach for an arbitrary normal random variable would require a separate table for every possible pair of values for μ and σ. Fortunately, all normal probability distributions are related algebraically, and Appendix Table II can be used to find the probabilities associated with an arbitrary normal random variable by first using a simple transformation.

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma} \qquad (4\text{-}10)$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a standard normal random variable.

Creating a new random variable by this transformation is referred to as **standardizing.** The random variable Z represents the distance of X from its mean in terms of standard deviations. It is the key step to calculate a probability for an arbitrary normal random variable.

EXAMPLE 4-13

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 $(\text{milliamperes})^2$. What is the probability that a measurement will exceed 13 milliamperes?

Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. Let $Z = (X - 10)/2$. The relationship between the several values of X and the transformed values of Z are shown in Fig. 4-15. We note that $X > 13$ corresponds to $Z > 1.5$. Therefore, from Appendix Table II,

$$P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681$$

Rather than using Fig. 4-15, the probability can be found from the inequality $X > 13$. That is,

$$P(X > 13) = P\left(\frac{(X - 10)}{2} > \frac{(13 - 10)}{2}\right) = P(Z > 1.5) = 0.06681$$

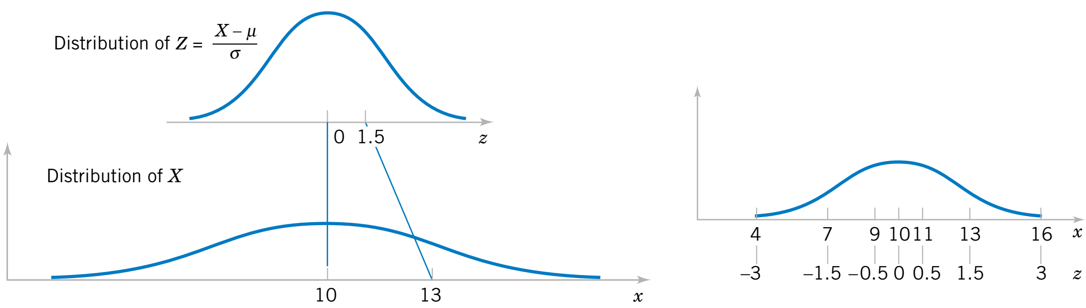


Figure 4-15 Standardizing a normal random variable.

In the preceding example, the value 13 is transformed to 1.5 by standardizing, and 1.5 is often referred to as the ***z*-value** associated with a probability. The following summarizes the calculation of probabilities derived from normal random variables.

> Suppose X is a normal random variable with mean μ and variance σ^2. Then,
>
> $$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P(Z \le z) \tag{4-11}$$
>
> where Z is a **standard normal random variable,** and $z = \dfrac{(x-\mu)}{\sigma}$ is the ***z*-value** obtained by **standardizing** X.
>
> The probability is obtained by entering Appendix Table II with $z = (x - \mu)/\sigma$.

EXAMPLE 4-14

Continuing the previous example, what is the probability that a current measurement is between 9 and 11 milliamperes? From Fig. 4-15, or by proceeding algebraically, we have

$$\begin{aligned} P(9 < X < 11) &= P((9-10)/2 < (X-10)/2 < (11-10)/2) \\ &= P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5) \\ &= 0.69146 - 0.30854 = 0.38292 \end{aligned}$$

Determine the value for which the probability that a current measurement is below this value is 0.98. The requested value is shown graphically in Fig. 4-16. We need the value of x such that $P(X < x) = 0.98$. By standardizing, this probability expression can be written as

$$\begin{aligned} P(X < x) &= P((X-10)/2 < (x-10)/2) \\ &= P(Z < (x-10)/2) \\ &= 0.98 \end{aligned}$$

Appendix Table II is used to find the z-value such that $P(Z < z) = 0.98$. The nearest probability from Table II results in

$$P(Z < 2.05) = 0.97982$$

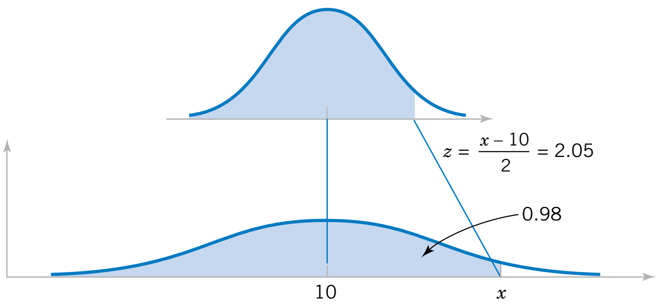


Figure 4-16 Determining the value of x to meet a specified probability.

Therefore, $(x - 10)/2 = 2.05$, and the standardizing transformation is used in reverse to solve for x. The result is

$$x = 2(2.05) + 10 = 14.1 \text{ milliamperes}$$

EXAMPLE 4-15 Assume that in the detection of a digital signal the background noise follows a normal distribution with a mean of 0 volt and standard deviation of 0.45 volt. The system assumes a digital 1 has been transmitted when the voltage exceeds 0.9. What is the probability of detecting a digital 1 when none was sent?

Let the random variable N denote the voltage of noise. The requested probability is

$$P(N > 0.9) = P\left(\frac{N}{0.45} > \frac{0.9}{0.45}\right) = P(Z > 2) = 1 - 0.97725 = 0.02275$$

This probability can be described as the probability of a false detection.

Determine symmetric bounds about 0 that include 99% of all noise readings. The question requires us to find x such that $P(-x < N < x) = 0.99$. A graph is shown in Fig. 4-17. Now,

$$\begin{aligned} P(-x < N < x) &= P(-x/0.45 < N/0.45 < x/0.45) \\ &= P(-x/0.45 < Z < x/0.45) = 0.99 \end{aligned}$$

From Appendix Table II

$$P(-2.58 < Z < 2.58) = 0.99$$

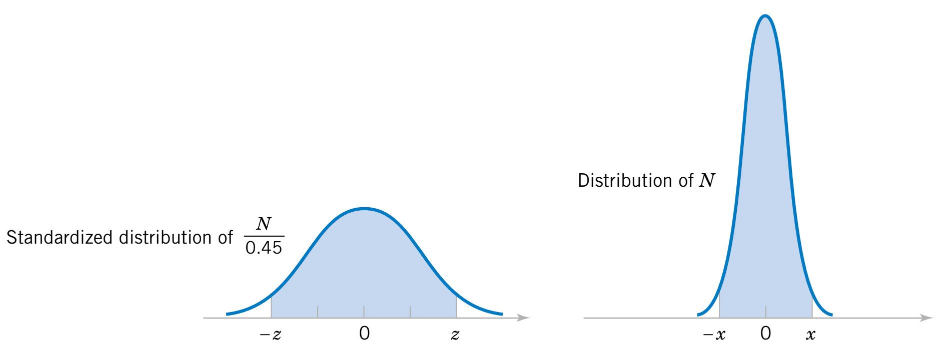


Figure 4-17 Determining the value of x to meet a specified probability.

Therefore,

$$x/0.45 = 2.58$$

and

$$x = 2.58(0.45) = 1.16$$

Suppose a digital 1 is represented as a shift in the mean of the noise distribution to 1.8 volts. What is the probability that a digital 1 is not detected? Let the random variable S denote the voltage when a digital 1 is transmitted. Then,

$$P(S < 0.9) = P\left(\frac{S - 1.8}{0.45} < \frac{0.9 - 1.8}{0.45}\right) = P(Z < -2) = 0.02275$$

This probability can be interpreted as the probability of a missed signal.

EXAMPLE 4-16

The diameter of a shaft in an optical storage drive is normally distributed with mean 0.2508 inch and standard deviation 0.0005 inch. The specifications on the shaft are 0.2500 ± 0.0015 inch. What proportion of shafts conforms to specifications?

Let X denote the shaft diameter in inches. The requested probability is shown in Fig. 4-18 and

$$\begin{aligned} P(0.2485 < X < 0.2515) &= P\left(\frac{0.2485 - 0.2508}{0.0005} < Z < \frac{0.2515 - 0.2508}{0.0005}\right) \\ &= P(-4.6 < Z < 1.4) = P(Z < 1.4) - P(Z < -4.6) \\ &= 0.91924 - 0.0000 = 0.91924 \end{aligned}$$

Most of the nonconforming shafts are too large, because the process mean is located very near to the upper specification limit. If the process is centered so that the process mean is equal to the target value of 0.2500,

$$\begin{aligned} P(0.2485 < X < 0.2515) &= P\left(\frac{0.2485 - 0.2500}{0.0005} < Z < \frac{0.2515 - 0.2500}{0.0005}\right) \\ &= P(-3 < Z < 3) \\ &= P(Z < 3) - P(Z < -3) \\ &= 0.99865 - 0.00135 \\ &= 0.9973 \end{aligned}$$

By recentering the process, the yield is increased to approximately 99.73%.

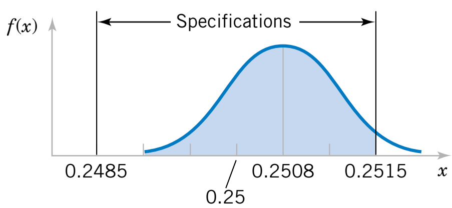


Figure 4-18 Distribution for Example 4-16.

Mean and Variance of the Normal Distribution (CD Only)

EXERCISES FOR SECTION 4-6

4-39. Use Appendix Table II to determine the following probabilities for the standard normal random variable Z:
(a) $P(Z < 1.32)$ (b) $P(Z < 3.0)$
(c) $P(Z > 1.45)$ (d) $P(Z > -2.15)$
(e) $P(-2.34 < Z < 1.76)$

4-40. Use Appendix Table II to determine the following probabilities for the standard normal random variable Z:
(a) $P(-1 < Z < 1)$ (b) $P(-2 < Z < 2)$
(c) $P(-3 < Z < 3)$ (d) $P(Z > 3)$
(e) $P(0 < Z < 1)$

4-41. Assume Z has a standard normal distribution. Use Appendix Table II to determine the value for z that solves each of the following:
(a) $P(Z < z) = 0.9$ (b) $P(Z < z) = 0.5$
(c) $P(Z > z) = 0.1$ (d) $P(Z > z) = 0.9$
(e) $P(-1.24 < Z < z) = 0.8$

4-42. Assume Z has a standard normal distribution. Use Appendix Table II to determine the value for z that solves each of the following:
(a) $P(-z < Z < z) = 0.95$ (b) $P(-z < Z < z) = 0.99$
(c) $P(-z < Z < z) = 0.68$ (d) $P(-z < Z < z) = 0.9973$

4-43. Assume X is normally distributed with a mean of 10 and a standard deviation of 2. Determine the following:
(a) $P(X < 13)$ (b) $P(X > 9)$
(c) $P(6 < X < 14)$ (d) $P(2 < X < 4)$
(e) $P(-2 < X < 8)$

4-44. Assume X is normally distributed with a mean of 10 and a standard deviation of 2. Determine the value for x that solves each of the following:
(a) $P(X > x) = 0.5$
(b) $P(X > x) = 0.95$
(c) $P(x < X < 10) = 0.2$
(d) $P(-x < X - 10 < x) = 0.95$
(e) $P(-x < X - 10 < x) = 0.99$

4-45. Assume X is normally distributed with a mean of 5 and a standard deviation of 4. Determine the following:
(a) $P(X < 11)$ (b) $P(X > 0)$
(c) $P(3 < X < 7)$ (d) $P(-2 < X < 9)$
(e) $P(2 < X < 8)$

4-46. Assume X is normally distributed with a mean of 5 and a standard deviation of 4. Determine the value for x that solves each of the following:
(a) $P(X > x) = 0.5$ (b) $P(X > x) = 0.95$
(c) $P(x < X < 9) = 0.2$ (d) $P(3 < X < x) = 0.95$
(e) $P(-x < X < x) = 0.99$

4-47. The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter.
(a) What is the probability that a sample's strength is less than 6250 Kg/cm^2?
(b) What is the probability that a sample's strength is between 5800 and 5900 Kg/cm^2?
(c) What strength is exceeded by 95% of the samples?

4-48. The tensile strength of paper is modeled by a normal distribution with a mean of 35 pounds per square inch and a standard deviation of 2 pounds per square inch.
(a) What is the probability that the strength of a sample is less than 40 lb/in^2?
(b) If the specifications require the tensile strength to exceed 30 lb/in^2, what proportion of the samples is scrapped?

4-49. The line width of for semiconductor manufacturing is assumed to be normally distributed with a mean of 0.5 micrometer and a standard deviation of 0.05 micrometer.
(a) What is the probability that a line width is greater than 0.62 micrometer?
(b) What is the probability that a line width is between 0.47 and 0.63 micrometer?
(c) The line width of 90% of samples is below what value?

4-50. The fill volume of an automated filling machine used for filling cans of carbonated beverage is normally distributed with a mean of 12.4 fluid ounces and a standard deviation of 0.1 fluid ounce.
(a) What is the probability a fill volume is less than 12 fluid ounces?
(b) If all cans less than 12.1 or greater than 12.6 ounces are scrapped, what proportion of cans is scrapped?
(c) Determine specifications that are symmetric about the mean that include 99% of all cans.

4-51. The time it takes a cell to divide (called mitosis) is normally distributed with an average time of one hour and a standard deviation of 5 minutes.
(a) What is the probability that a cell divides in less than 45 minutes?
(b) What is the probability that it takes a cell more than 65 minutes to divide?
(c) What is the time that it takes approximately 99% of all cells to complete mitosis?

4-52. In the previous exercise, suppose that the mean of the filling operation can be adjusted easily, but the standard deviation remains at 0.1 ounce.
(a) At what value should the mean be set so that 99.9% of all cans exceed 12 ounces?
(b) At what value should the mean be set so that 99.9% of all cans exceed 12 ounces if the standard deviation can be reduced to 0.05 fluid ounce?

4-53. The reaction time of a driver to visual stimulus is normally distributed with a mean of 0.4 seconds and a standard deviation of 0.05 seconds.
(a) What is the probability that a reaction requires more than 0.5 seconds?
(b) What is the probability that a reaction requires between 0.4 and 0.5 seconds?
(c) What is the reaction time that is exceeded 90% of the time?

4-54. The speed of a file transfer from a server on campus to a personal computer at a student's home on a weekday evening is normally distributed with a mean of 60 kilobits per second and a standard deviation of 4 kilobits per second.
(a) What is the probability that the file will transfer at a speed of 70 kilobits per second or more?
(b) What is the probability that the file will transfer at a speed of less than 58 kilobits per second?
(c) If the file is 1 megabyte, what is the average time it will take to transfer the file? (Assume eight bits per byte.)

4-55. The length of an injection-molded plastic case that holds magnetic tape is normally distributed with a length of 90.2 millimeters and a standard deviation of 0.1 millimeter.
(a) What is the probability that a part is longer than 90.3 millimeters or shorter than 89.7 millimeters?
(b) What should the process mean be set at to obtain the greatest number of parts between 89.7 and 90.3 millimeters?
(c) If parts that are not between 89.7 and 90.3 millimeters are scrapped, what is the yield for the process mean that you selected in part (b)?

4-56. In the previous exercise assume that the process is centered so that the mean is 90 millimeters and the standard deviation is 0.1 millimeter. Suppose that 10 cases are measured, and they are assumed to be independent.
(a) What is the probability that all 10 cases are between 89.7 and 90.3 millimeters?
(b) What is the expected number of the 10 cases that are between 89.7 and 90.3 millimeters?

4-57. The sick-leave time of employees in a firm in a month is normally distributed with a mean of 100 hours and a standard deviation of 20 hours.
(a) What is the probability that the sick-leave time for next month will be between 50 and 80 hours?
(b) How much time should be budgeted for sick leave if the budgeted amount should be exceeded with a probability of only 10%?

4-58. The life of a semiconductor laser at a constant power is normally distributed with a mean of 7000 hours and a standard deviation of 600 hours.
(a) What is the probability that a laser fails before 5000 hours?
(b) What is the life in hours that 95% of the lasers exceed?
(c) If three lasers are used in a product and they are assumed to fail independently, what is the probability that all three are still operating after 7000 hours?

4-59. The diameter of the dot produced by a printer is normally distributed with a mean diameter of 0.002 inch and a standard deviation of 0.0004 inch.
(a) What is the probability that the diameter of a dot exceeds 0.0026 inch?
(b) What is the probability that a diameter is between 0.0014 and 0.0026 inch?
(c) What standard deviation of diameters is needed so that the probability in part (b) is 0.995?

4-60. The weight of a sophisticated running shoe is normally distributed with a mean of 12 ounces and a standard deviation of 0.5 ounce.
(a) What is the probability that a shoe weighs more than 13 ounces?
(b) What must the standard deviation of weight be in order for the company to state that 99.9% of its shoes are less than 13 ounces?
(c) If the standard deviation remains at 0.5 ounce, what must the mean weight be in order for the company to state that 99.9% of its shoes are less than 13 ounces?

4-7 NORMAL APPROXIMATION TO THE BINOMIAL AND POISSON DISTRIBUTIONS

We began our section on the normal distribution with the central limit theorem and the normal distribution as an approximation to a random variable with a large number of trials. Consequently, it should not be a surprise to learn that the normal distribution can be used to approximate binomial probabilities for cases in which n is large. The following example illustrates that for many physical systems the binomial model is appropriate with an extremely large value for n. In these cases, it is difficult to calculate probabilities by using the binomial distribution. Fortunately, the normal approximation is most effective in these cases. An illustration is provided in Fig. 4-19. The area of each bar equals the binomial probability of x. Notice that the area of bars can be approximated by areas under the normal density function.

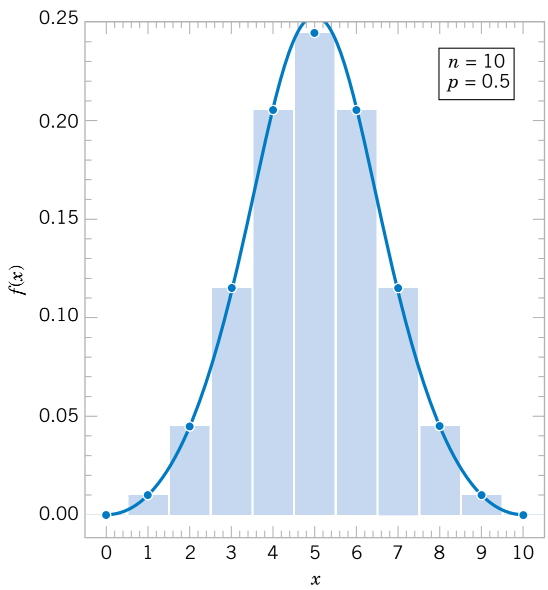


Figure 4-19 Normal approximation to the binomial distribution.

EXAMPLE 4-17 In a digital communication channel, assume that the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit is received in error is 1×10^{-5}. If 16 million bits are transmitted, what is the probability that more than 150 errors occur?

Let the random variable X denote the number of errors. Then X is a binomial random variable and

$$P(X > 150) = 1 - P(x \le 150) = 1 - \sum_{x=0}^{150} \binom{16{,}000{,}000}{x} (10^{-5})^x (1 - 10^{-5})^{16{,}000{,}000-x}$$

Clearly, the probability in Example 4-17 is difficult to compute. Fortunately, the normal distribution can be used to provide an excellent approximation in this example.

Normal Approximation to the Binomial Distribution

If X is a binomial random variable,

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \tag{4-12}$$

is approximately a standard normal random variable. The approximation is good for

$$np > 5 \quad \text{and} \quad n(1 - p) > 5$$

Recall that for a binomial variable X, $E(X) = np$ and $V(X) = np(1 - p)$. Consequently, the expression in Equation 4-12 is nothing more than the formula for standardizing the random variable X. Probabilities involving X can be approximated by using a standard normal distribution. The approximation is good when n is large relative to p.

EXAMPLE 4-18 The digital communication problem in the previous example is solved as follows:

$$P(X > 150) = P\left(\frac{X - 160}{\sqrt{160(1 - 10^{-5})}} > \frac{150 - 160}{\sqrt{160(1 - 10^{-5})}}\right)$$
$$= P(Z > -0.79) = P(Z < 0.79) = 0.785$$

Because $np = (16 \times 10^6)(1 \times 10^{-5}) = 160$ and $n(1 - p)$ is much larger, the approximation is expected to work well in this case.

EXAMPLE 4-19 Again consider the transmission of bits in Example 4-18. To judge how well the normal approximation works, assume only $n = 50$ bits are to be transmitted and that the probability of an error is $p = 0.1$. The exact probability that 2 or less errors occur is

$$P(X \leq 2) = \binom{50}{0} 0.9^{50} + \binom{50}{1} 0.1(0.9^{49}) + \binom{50}{2} 0.1^2(0.9^{48}) = 0.112$$

Based on the normal approximation

$$P(X \leq 2) = P\left(\frac{X - 5}{2.12} < \frac{2 - 5}{2.12}\right) = P(Z < -1.42) = 0.08$$

Even for a sample as small as 50 bits, the normal approximation is reasonable.

If np or $n(1 - p)$ is small, the binomial distribution is quite skewed and the symmetric normal distribution is not a good approximation. Two cases are illustrated in Fig. 4-20. However, a correction factor can be used that will further improve the approximation. This factor is called a **continuity correction** and it is discussed in Section 4-8 on the CD.

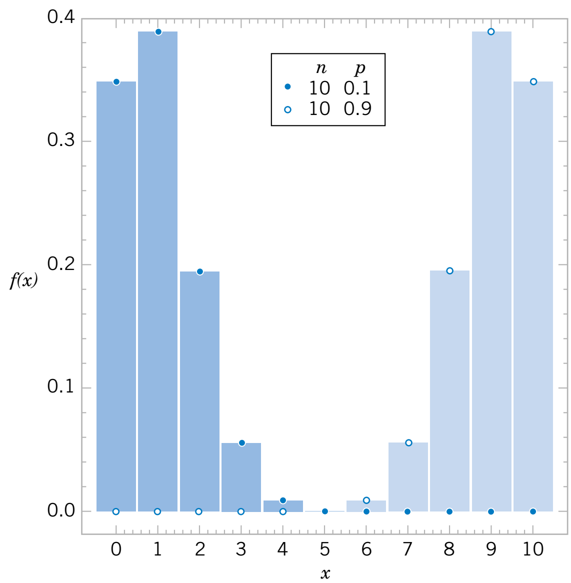


Figure 4-20 Binomial distribution is not symmetrical if p is near 0 or 1.

$$\text{hypergometric distribution} \quad \underset{\frac{n}{N} < 0.1}{\approx} \quad \text{binomial distribution} \quad \underset{\substack{np > 5 \\ n(1-p) > 5}}{\approx} \quad \text{normal distribution}$$

Figure 4-21 Conditions for approximating hypergeometric and binomial probabilities.

Recall that the binomial distribution is a satisfactory approximation to the hypergeometric distribution when n, the sample size, is small relative to N, the size of the population from which the sample is selected. A rule of thumb is that the binomial approximation is effective if $n/N < 0.1$. Recall that for a hypergeometric distribution p is defined as $p = K/N$. That is, p is interpreted as the number of successes in the population. Therefore, the normal distribution can provide an effective approximation of hypergeometric probabilities when $n/N < 0.1$, $np > 5$ and $n(1 - p) > 5$. Figure 4-21 provides a summary of these guidelines.

Recall that the Poisson distribution was developed as the limit of a binomial distribution as the number of trials increased to infinity. Consequently, it should not be surprising to find that the normal distribution can also be used to approximate probabilities of a Poisson random variable.

Normal Approximation to the Poisson Distribution

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \tag{4-13}$$

is approximately a standard normal random variable. The approximation is good for

$$\lambda > 5$$

EXAMPLE 4-20

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that less than 950 particles are found?

This probability can be expressed exactly as

$$P(X \le 950) = \sum_{x=0}^{950} \frac{e^{-1000} x^{1000}}{x!}$$

The computational difficulty is clear. The probability can be approximated as

$$P(X \le x) = P\left(Z \le \frac{950 - 1000}{\sqrt{1000}}\right) = P(Z \le -1.58) = 0.057$$

EXERCISES FOR SECTION 4-7

4-61. Suppose that X is a binomial random variable with $n = 200$ and $p = 0.4$.

(a) Approximate the probability that X is less than or equal to 70.

(b) Approximate the probability that X is greater than 70 and less than 90.

4-62. Suppose that X is a binomial random variable with $n = 100$ and $p = 0.1$.

(a) Compute the exact probability that X is less than 4.

(b) Approximate the probability that X is less than 4 and compare to the result in part (a).

(c) Approximate the probability that $8 < X < 12$.

4-63. The manufacturing of semiconductor chips produces 2% defective chips. Assume the chips are independent and that a lot contains 1000 chips.
(a) Approximate the probability that more than 25 chips are defective.
(b) Approximate the probability that between 20 and 30 chips are defective.

4-64. A supplier ships a lot of 1000 electrical connectors. A sample of 25 is selected at random, without replacement. Assume the lot contains 100 defective connectors.
(a) Using a binomial approximation, what is the probability that there are no defective connectors in the sample?
(b) Use the normal approximation to answer the result in part (a). Is the approximation satisfactory?
(c) Redo parts (a) and (b) assuming the lot size is 500. Is the normal approximation to the probability that there are no defective connectors in the sample satisfactory in this case?

4-65. An electronic office product contains 5000 electronic components. Assume that the probability that each component operates without failure during the useful life of the product is 0.999, and assume that the components fail independently. Approximate the probability that 10 or more of the original 5000 components fail during the useful life of the product.

4-66. Suppose that the number of asbestos particles in a sample of 1 squared centimeter of dust is a Poisson random variable with a mean of 1000. What is the probability that 10 squared centimeters of dust contains more than 10,000 particles?

4-67. A corporate Web site contains errors on 50 of 1000 pages. If 100 pages are sampled randomly, without replacement, approximate the probability that at least 1 of the pages in error are in the sample.

4-68. Hits to a high-volume Web site are assumed to follow a Poisson distribution with a mean of 10,000 per day. Approximate each of the following:
(a) The probability of more than 20,000 hits in a day
(b) The probability of less than 9900 hits in a day
(c) The value such that the probability that the number of hits in a day exceed the value is 0.01

4-69. Continuation of Exercise 4-68.
(a) Approximate the expected number of days in a year (365 days) that exceed 10,200 hits.
(b) Approximate the probability that over a year (365 days) more than 15 days each have more than 10,200 hits.

4-70. The percentage of people exposed to a bacteria who become ill is 20%. Assume that people are independent. Assume that 1000 people are exposed to the bacteria. Approximate each of the following:
(a) The probability that more than 225 become ill
(b) The probability that between 175 and 225 become ill
(c) The value such that the probability that the number of people that become ill exceeds the value is 0.01

4-71. A high-volume printer produces minor print-quality errors on a test pattern of 1000 pages of text according to a Poisson distribution with a mean of 0.4 per page.
(a) Why are the number of errors on each page independent random variables?
(b) What is the mean number of pages with errors (one or more)?
(c) Approximate the probability that more than 350 pages contain errors (one or more).

4-8 CONTINUITY CORRECTION TO IMPROVE THE APPROXIMATION (CD ONLY)

4-9 EXPONENTIAL DISTRIBUTION

The discussion of the Poisson distribution defined a random variable to be the number of flaws along a length of copper wire. The distance between flaws is another random variable that is often of interest. Let the random variable X denote the length from any starting point on the wire until a flaw is detected.

As you might expect, the distribution of X can be obtained from knowledge of the distribution of the number of flaws. The key to the relationship is the following concept. The distance to the first flaw exceeds 3 millimeters if and only if there are no flaws within a length of 3 millimeters—simple, but sufficient for an analysis of the distribution of X.

In general, let the random variable N denote the number of flaws in x millimeters of wire. If the mean number of flaws is λ per millimeter, N has a Poisson distribution with mean λx. We assume that the wire is longer than the value of x. Now,

$$P(X > x) = P(N = 0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = e^{-\lambda x}$$

Therefore,

$$F(x) = P(X \le x) = 1 - e^{-\lambda x}, \qquad x \ge 0$$

is the cumulative distribution function of X. By differentiating $F(x)$, the probability density function of X is calculated to be

$$f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$$

The derivation of the distribution of X depends only on the assumption that the flaws in the wire follow a **Poisson process.** Also, the starting point for measuring X doesn't matter because the probability of the number of flaws in an interval of a Poisson process depends only on the length of the interval, not on the location. For any Poisson process, the following general result applies.

Definition

The random variable X that equals the distance between successive counts of a Poisson process with mean $\lambda > 0$ is an **exponential random variable** with parameter λ. The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad 0 \le x < \infty \tag{4-14}$$

The exponential distribution obtains its name from the exponential function in the probability density function. Plots of the exponential distribution for selected values of λ are shown in Fig. 4-22. For any value of λ, the exponential distribution is quite skewed. The following results are easily obtained and are left as an exercise.

If the random variable X has an exponential distribution with parameter λ,

$$\mu = E(X) = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = V(X) = \frac{1}{\lambda^2} \tag{4-15}$$

It is important to **use consistent units** in the calculation of probabilities, means, and variances involving exponential random variables. The following example illustrates unit conversions.

EXAMPLE 4-21

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour. What is the probability that there are no log-ons in an interval of 6 minutes?

Let X denote the time in hours from the start of the interval until the first log-on. Then, X has an exponential distribution with $\lambda = 25$ log-ons per hour. We are interested in the probability that X exceeds 6 minutes. Because λ is given in log-ons per hour, we express all time units in hours. That is, 6 minutes $= 0.1$ hour. The probability requested is shown as the shaded area under the probability density function in Fig. 4-23. Therefore,

$$P(X > 0.1) = \int_{0.1}^{\infty} 25e^{-25x}\,dx = e^{-25(0.1)} = 0.082$$

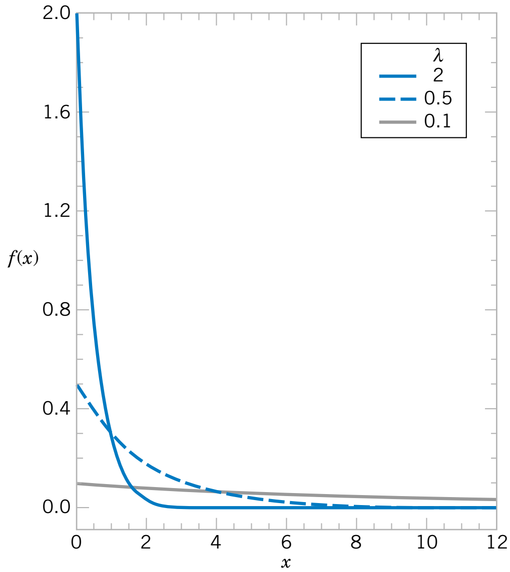


Figure 4-22 Probability density function of exponential random variables for selected values of λ.

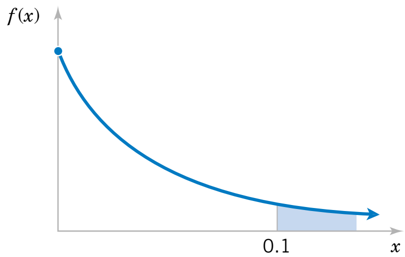


Figure 4-23 Probability for the exponential distribution in Example 4-21.

Also, the cumulative distribution function can be used to obtain the same result as follows:

$$P(X > 0.1) = 1 - F(0.1) = e^{-25(0.1)}$$

An identical answer is obtained by expressing the mean number of log-ons as 0.417 log-ons per minute and computing the probability that the time until the next log-on exceeds 6 minutes. Try it.

What is the probability that the time until the next log-on is between 2 and 3 minutes? Upon converting all units to hours,

$$P(0.033 < X < 0.05) = \int_{0.033}^{0.05} 25e^{-25x}\,dx = -e^{-25x}\Big|_{0.033}^{0.05} = 0.152$$

An alternative solution is

$$P(0.033 < X < 0.05) = F(0.05) - F(0.033) = 0.152$$

Determine the interval of time such that the probability that no log-on occurs in the interval is 0.90. The question asks for the length of time x such that $P(X > x) = 0.90$. Now,

$$P(X > x) = e^{-25x} = 0.90$$

Take the (natural) log of both sides to obtain $-25x = \ln(0.90) = -0.1054$. Therefore,

$$x = 0.00421 \text{ hour} = 0.25 \text{ minute}$$

Furthermore, the mean time until the next log-on is

$$\mu = 1/25 = 0.04 \text{ hour} = 2.4 \text{ minutes}$$

The standard deviation of the time until the next log-on is

$$\sigma = 1/25 \text{ hours} = 2.4 \text{ minutes}$$

In the previous example, the probability that there are no log-ons in a 6-minute interval is 0.082 regardless of the starting time of the interval. A Poisson process assumes that events occur uniformly throughout the interval of observation; that is, there is no clustering of events. If the log-ons are well modeled by a Poisson process, the probability that the first log-on after noon occurs after 12:06 P.M. is the same as the probability that the first log-on after 3:00 P.M. occurs after 3:06 P.M. And if someone logs on at 2:22 P.M., the probability the next log-on occurs after 2:28 P.M. is still 0.082.

Our starting point for observing the system does not matter. However, if there are high-use periods during the day, such as right after 8:00 A.M., followed by a period of low use, a Poisson process is not an appropriate model for log-ons and the distribution is not appropriate for computing probabilities. It might be reasonable to model each of the high- and low-use periods by a separate Poisson process, employing a larger value for λ during the high-use periods and a smaller value otherwise. Then, an exponential distribution with the corresponding value of λ can be used to calculate log-on probabilities for the high- and low-use periods.

Lack of Memory Property

An even more interesting property of an exponential random variable is concerned with conditional probabilities.

EXAMPLE 4-22

Let X denote the time between detections of a particle with a geiger counter and assume that X has an exponential distribution with $\lambda = 1.4$ minutes. The probability that we detect a particle within 30 seconds of starting the counter is

$$P(X < 0.5 \text{ minute}) = F(0.5) = 1 - e^{-0.5/1.4} = 0.30$$

In this calculation, all units are converted to minutes. Now, suppose we turn on the geiger counter and wait 3 minutes without detecting a particle. What is the probability that a particle is detected in the next 30 seconds?

Because we have already been waiting for 3 minutes, we feel that we are "due." That is, the probability of a detection in the next 30 seconds should be greater than 0.3. However, for an exponential distribution, this is not true. The requested probability can be expressed as the conditional probability that $P(X < 3.5 \mid X > 3)$. From the definition of conditional probability,

$$P(X < 3.5 \mid X > 3) = P(3 < X < 3.5)/P(X > 3)$$

where

$$P(3 < X < 3.5) = F(3.5) - F(3) = [1 - e^{-3.5/1.4}] - [1 - e^{-3/1.4}] = 0.0035$$

and

$$P(X > 3) = 1 - F(3) = e^{-3/1.4} = 0.117$$

Therefore,

$$P(X < 3.5 \mid X > 3) = 0.035/0.117 = 0.30$$

After waiting for 3 minutes without a detection, the probability of a detection in the next 30 seconds is the same as the probability of a detection in the 30 seconds immediately after starting the counter. The fact that you have waited 3 minutes without a detection does not change the probability of a detection in the next 30 seconds.

Example 4-22 illustrates the **lack of memory property** of an exponential random variable and a general statement of the property follows. In fact, the exponential distribution is the only continuous distribution with this property.

Lack of Memory Property

For an exponential random variable X,

$$P(X < t_1 + t_2 \mid X > t_1) = P(X < t_2) \quad (4\text{-}16)$$

Figure 4-24 graphically illustrates the lack of memory property. The area of region A divided by the total area under the probability density function $(A + B + C + D = 1)$ equals $P(X < t_2)$. The area of region C divided by the area $C + D$ equals $P(X < t_1 + t_2 \mid X > t_1)$. The lack of memory property implies that the proportion of the total area that is in A equals the proportion of the area in C and D that is in C. The mathematical verification of the lack of memory property is left as a mind-expanding exercise.

The lack of memory property is not that surprising when you consider the development of a Poisson process. In that development, we assumed that an interval could be partitioned into small intervals that were independent. These subintervals are similar to independent Bernoulli trials that comprise a binomial process; knowledge of previous results does not affect the probabilities of events in future subintervals. An exponential random variable is the continuous analog of a geometric random variable, and they share a similar lack of memory property.

The exponential distribution is often used in reliability studies as the model for the time until failure of a device. For example, the lifetime of a semiconductor chip might be modeled as an exponential random variable with a mean of 40,000 hours. The lack of

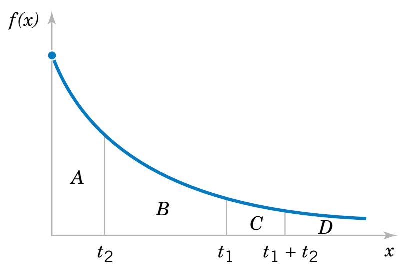


Figure 4-24 Lack of memory property of an exponential distribution.

memory property of the exponential distribution implies that the device does not wear out. That is, regardless of how long the device has been operating, the probability of a failure in the next 1000 hours is the same as the probability of a failure in the first 1000 hours of operation. The lifetime L of a device with failures caused by random shocks might be appropriately modeled as an exponential random variable. However, the lifetime L of a device that suffers slow mechanical wear, such as bearing wear, is better modeled by a distribution such that $P(L < t + \Delta t | L > t)$ increases with t. Distributions such as the Weibull distribution are often used, in practice, to model the failure time of this type of device. The Weibull distribution is presented in a later section.

EXERCISES FOR SECTION 4-9

4-72. Suppose X has an exponential distribution with $\lambda = 2$. Determine the following:
(a) $P(X \leq 0)$ (b) $P(X \geq 2)$
(c) $P(X \leq 1)$ (d) $P(1 < X < 2)$
(e) Find the value of x such that $P(X < x) = 0.05$.

4-73. Suppose X has an exponential distribution with mean equal to 10. Determine the following:
(a) $P(X > 10)$
(b) $P(X > 20)$
(c) $P(X > 30)$
(d) Find the value of x such that $P(X < x) = 0.95$.

4-74. Suppose the counts recorded by a geiger counter follow a Poisson process with an average of two counts per minute.
(a) What is the probability that there are no counts in a 30-second interval?
(b) What is the probability that the first count occurs in less than 10 seconds?
(c) What is the probability that the first count occurs between 1 and 2 minutes after start-up?

4-75. Suppose that the log-ons to a computer network follow a Poisson process with an average of 3 counts per minute.
(a) What is the mean time between counts?
(b) What is the standard deviation of the time between counts?
(c) Determine x such that the probability that at least one count occurs before time x minutes is 0.95.

4-76. The time to failure (in hours) for a laser in a cytometry machine is modeled by an exponential distribution with $\lambda = 0.00004$.
(a) What is the probability that the laser will last at least 20,000 hours?
(b) What is the probability that the laser will last at most 30,000 hours?
(c) What is the probability that the laser will last between 20,000 and 30,000 hours?

4-77. The time between calls to a plumbing supply business is exponentially distributed with a mean time between calls of 15 minutes.
(a) What is the probability that there are no calls within a 30-minute interval?
(b) What is the probability that at least one call arrives within a 10-minute interval?
(c) What is the probability that the first call arrives within 5 and 10 minutes after opening?
(d) Determine the length of an interval of time such that the probability of at least one call in the interval is 0.90.

4-78. The life of automobile voltage regulators has an exponential distribution with a mean life of six years. You purchase an automobile that is six years old, with a working voltage regulator, and plan to own it for six years.
(a) What is the probability that the voltage regulator fails during your ownership?
(b) If your regulator fails after you own the automobile three years and it is replaced, what is the mean time until the next failure?

4-79. The time to failure (in hours) of fans in a personal computer can be modeled by an exponential distribution with $\lambda = 0.0003$.
(a) What proportion of the fans will last at least 10,000 hours?
(b) What proportion of the fans will last at most 7000 hours?

4-80. The time between the arrival of electronic messages at your computer is exponentially distributed with a mean of two hours.
(a) What is the probability that you do not receive a message during a two-hour period?
(b) If you have not had a message in the last four hours, what is the probability that you do not receive a message in the next two hours?
(c) What is the expected time between your fifth and sixth messages?

4-81. The time between arrivals of taxis at a busy intersection is exponentially distributed with a mean of 10 minutes.
(a) What is the probability that you wait longer than one hour for a taxi?
(b) Suppose you have already been waiting for one hour for a taxi, what is the probability that one arrives within the next 10 minutes?

4-82. Continuation of Exercise 4-81.
(a) Determine x such that the probability that you wait more than x minutes is 0.10.

(b) Determine x such that the probability that you wait less than x minutes is 0.90.
(c) Determine x such that the probability that you wait less than x minutes is 0.50.

4-83. The distance between major cracks in a highway follows an exponential distribution with a mean of 5 miles.
(a) What is the probability that there are no major cracks in a 10-mile stretch of the highway?
(b) What is the probability that there are two major cracks in a 10-mile stretch of the highway?
(c) What is the standard deviation of the distance between major cracks?

4-84. Continuation of Exercise 4-83.
(a) What is the probability that the first major crack occurs between 12 and 15 miles of the start of inspection?
(b) What is the probability that there are no major cracks in two separate 5-mile stretches of the highway?
(c) Given that there are no cracks in the first 5 miles inspected, what is the probability that there are no major cracks in the next 10 miles inspected?

4-85. The lifetime of a mechanical assembly in a vibration test is exponentially distributed with a mean of 400 hours.
(a) What is the probability that an assembly on test fails in less than 100 hours?
(b) What is the probability that an assembly operates for more than 500 hours before failure?
(c) If an assembly has been on test for 400 hours without a failure, what is the probability of a failure in the next 100 hours?

4-86. Continuation of Exercise 4-85.
(a) If 10 assemblies are tested, what is the probability that at least one fails in less than 100 hours? Assume that the assemblies fail independently.
(b) If 10 assemblies are tested, what is the probability that all have failed by 800 hours? Assume the assemblies fail independently.

4-87. When a bus service reduces fares, a particular trip from New York City to Albany, New York, is very popular. A small bus can carry four passengers. The time between calls for tickets is exponentially distributed with a mean of 30 minutes. Assume that each call orders one ticket. What is the probability that the bus is filled in less than 3 hours from the time of the fare reduction?

4-88. The time between arrivals of small aircraft at a county airport is exponentially distributed with a mean of one hour. What is the probability that more than three aircraft arrive within an hour?

4-89. Continuation of Exercise 4-88.
(a) If 30 separate one-hour intervals are chosen, what is the probability that no interval contains more than three arrivals?
(b) Determine the length of an interval of time (in hours) such that the probability that no arrivals occur during the interval is 0.10.

4-90. The time between calls to a corporate office is exponentially distributed with a mean of 10 minutes.
(a) What is the probability that there are more than three calls in one-half hour?
(b) What is the probability that there are no calls within one-half hour?
(c) Determine x such that the probability that there are no calls within x hours is 0.01.

4-91. Continuation of Exercise 4-90.
(a) What is the probability that there are no calls within a two-hour interval?
(b) If four nonoverlapping one-half hour intervals are selected, what is the probability that none of these intervals contains any call?
(c) Explain the relationship between the results in part (a) and (b).

4-92. If the random variable X has an exponential distribution with mean θ, determine the following:
(a) $P(X > \theta)$ (b) $P(X > 2\theta)$
(c) $P(X > 3\theta)$
(d) How do the results depend on θ?

4-93. Assume that the flaws along a magnetic tape follow a Poisson distribution with a mean of 0.2 flaw per meter. Let X denote the distance between two successive flaws.
(a) What is the mean of X?
(b) What is the probability that there are no flaws in 10 consecutive meters of tape?
(c) Does your answer to part (b) change if the 10 meters are not consecutive?
(d) How many meters of tape need to be inspected so that the probability that at least one flaw is found is 90%?

4-94. Continuation of Exercise 4-93. (More difficult questions.)
(a) What is the probability that the first time the distance between two flaws exceeds 8 meters is at the fifth flaw?
(b) What is the mean number of flaws before a distance between two flaws exceeds 8 meters?

4-95. Derive the formula for the mean and variance of an exponential random variable.

4-10 ERLANG AND GAMMA DISTRIBUTIONS

4-10.1 Erlang Distribution

An exponential random variable describes the length until the first count is obtained in a Poisson process. A generalization of the exponential distribution is the length until r counts occur in a Poisson process. The random variable that equals the interval length until r counts occur in a Poisson process has an **Erlang random variable.**

EXAMPLE 4-23

The failures of the central processor units of large computer systems are often modeled as a Poisson process. Typically, failures are not caused by components wearing out, but by more random failures of the large number of semiconductor circuits in the units. Assume that the units that fail are immediately repaired, and assume that the mean number of failures per hour is 0.0001. Let X denote the time until four failures occur in a system. Determine the probability that X exceeds 40,000 hours.

Let the random variable N denote the number of failures in 40,000 hours of operation. The time until four failures occur exceeds 40,000 hours if and only if the number of failures in 40,000 hours is three or less. Therefore,

$$P(X > 40{,}000) = P(N \leq 3)$$

The assumption that the failures follow a Poisson process implies that N has a Poisson distribution with

$$E(N) = 40{,}000(0.0001) = 4 \text{ failures per 40,000 hours}$$

Therefore,

$$P(X > 40{,}000) = P(N \leq 3) = \sum_{k=0}^{3} \frac{e^{-4}4^k}{k!} = 0.433$$

The cumulative distribution function of a general Erlang random variable X can be obtained from $P(X \leq x) = 1 - P(X > x)$, and $P(X > x)$ can be determined as in the previous example. Then, the probability density function of X can be obtained by differentiating the cumulative distribution function and using a great deal of algebraic simplification. The details are left as an exercise. In general, we can obtain the following result.

Definition

The random variable X that equals the interval length until r counts occur in a Poisson process with mean $\lambda > 0$ has an **Erlang random variable** with parameters λ and r. The probability density function of X is

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}, \quad \text{for } x > 0 \text{ and } r = 1, 2, \ldots \tag{4-17}$$

Sketches of the Erlang probability density function for several values of r and λ are shown in Fig. 4-25. Clearly, an Erlang random variable with $r = 1$ is an exponential random variable. Probabilities involving Erlang random variables are often determined by computing a summation of Poisson random variables as in Example 4-23. The probability density function of an Erlang random variable can be used to determine probabilities; however, integrating by parts is often necessary. As was the case for the exponential distribution, one must be careful to define the random variable and the parameter in consistent units.

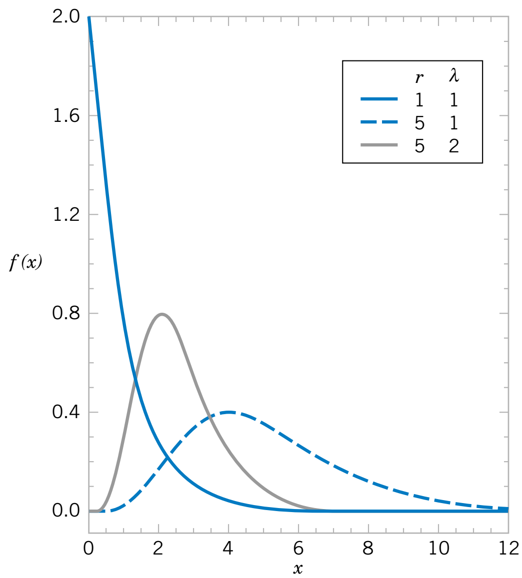


Figure 4-25 Erlang probability density functions for selected values of r and λ.

EXAMPLE 4-24 An alternative approach to computing the probability requested in Example 4-24 is to integrate the probability density function of X. That is,

$$P(X > 40{,}000) = \int_{40{,}000}^{\infty} f(x)\,dx = \int_{40{,}000}^{\infty} \frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}\,dx$$

where $r = 4$ and $\lambda = 0.0001$. Integration by parts can be used to verify the result obtained previously.

An Erlang random variable can be thought of as the continuous analog of a negative binomial random variable. A negative binomial random variable can be expressed as the sum of r geometric random variables. Similarly, an Erlang random variable can be represented as the sum of r exponential random variables. Using this conclusion, we can obtain the following plausible result. Sums of random variables are studied in Chapter 5.

If X is an Erlang random variable with parameters λ and r,

$$\mu = E(X) = r/\lambda \quad \text{and} \quad \sigma^2 = V(X) = r/\lambda^2 \qquad (4\text{-}18)$$

4-10.2 Gamma Distribution

The Erlang distribution is a special case of the **gamma distribution.** If the parameter r of an Erlang random variable is not an integer, but $r > 0$, the random variable has a gamma distribution. However, in the Erlang density function, the parameter r appears as r factorial.

Therefore, to define a gamma random variable, we require a generalization of the factorial function.

Definition

The **gamma function** is

$$\Gamma(r) = \int_0^{\infty} x^{r-1} e^{-x}\, dx, \quad \text{for } r > 0 \tag{4-19}$$

It can be shown that the integral in the definition of $\Gamma(r)$ is finite. Furthermore, by using integration by parts it can be shown that

$$\Gamma(r) = (r-1)\Gamma(r-1)$$

This result is left as an exercise. Therefore, if r is a positive integer (as in the Erlang distribution),

$$\Gamma(r) = (r-1)!$$

Also, $\Gamma(1) = 0! = 1$ and it can be shown that $\Gamma(1/2) = \pi^{1/2}$. The gamma function can be interpreted as a generalization to noninteger values of r of the term $(r-1)!$ that is used in the Erlang probability density function.

Now the gamma probability density function can be stated.

Definition

The random variable X with probability density function

$$f(x) = \frac{\lambda^r x^{r-1} e^{-\lambda x}}{\Gamma(r)}, \quad \text{for } x > 0 \tag{4-20}$$

has a **gamma random variable** with parameters $\lambda > 0$ and $r > 0$. If r is an integer, X has an Erlang distribution.

Sketches of the gamma distribution for several values of λ and r are shown in Fig. 4-26. It can be shown that $f(x)$ satisfies the properties of a probability density function, and the following result can be obtained. Repeated integration by parts can be used, but the details are lengthy.

If X is a **gamma random variable** with parameters λ and r,

$$\mu = E(X) = r/\lambda \quad \text{and} \quad \sigma^2 = V(X) = r/\lambda^2 \tag{4-21}$$

Although the gamma distribution is not frequently used as a model for a physical system, the special case of the Erlang distribution is very useful for modeling random experiments. The exercises provide illustrations. Furthermore, the **chi-squared distribution** is a special case of

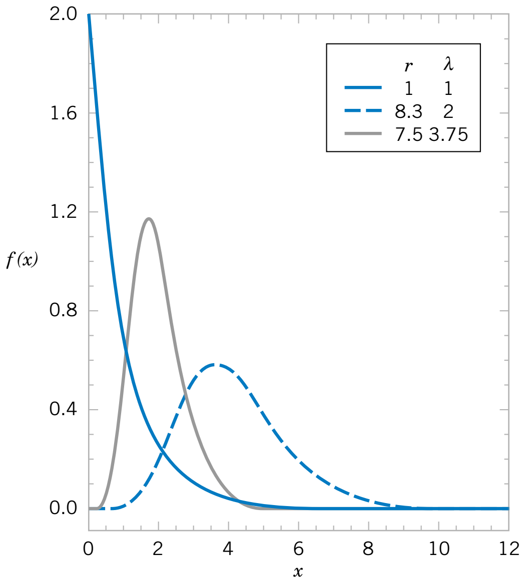


Figure 4-26 Gamma probability density functions for selected values of λ and r.

the gamma distribution in which $\lambda = 1/2$ and r equals one of the values $1/2, 1, 3/2, 2, \ldots$. This distribution is used extensively in interval estimation and tests of hypotheses that are discussed in subsequent chapters.

EXERCISES FOR SECTION 4-10

4-96. Calls to a telephone system follow a Poisson distribution with a mean of five calls per minute.
(a) What is the name applied to the distribution and parameter values of the time until the tenth call?
(b) What is the mean time until the tenth call?
(c) What is the mean time between the ninth and tenth calls?

4-97. Continuation of Exercise 4-96.
(a) What is the probability that exactly four calls occur within one minute?
(b) If 10 separate one-minute intervals are chosen, what is the probability that all intervals contain more than two calls?

4-98. Raw materials are studied for contamination. Suppose that the number of particles of contamination per pound of material is a Poisson random variable with a mean of 0.01 particle per pound.
(a) What is the expected number of pounds of material required to obtain 15 particles of contamination?
(b) What is the standard deviation of the pounds of materials required to obtain 15 particles of contamination?

4-99. The time between failures of a laser in a cytogenics machine is exponentially distributed with a mean of 25,000 hours.
(a) What is the expected time until the second failure?
(b) What is the probability that the time until the third failure exceeds 50,000 hours?

4-100. In a data communication system, several messages that arrive at a node are bundled into a packet before they are transmitted over the network. Assume the messages arrive at the node according to a Poisson process with $\tau = 30$ messages per minute. Five messages are used to form a packet.
(a) What is the mean time until a packet is formed, that is, until five messages arrived at the node?
(b) What is the standard deviation of the time until a packet is formed?
(c) What is the probability that a packet is formed in less than 10 seconds?
(d) What is the probability that a packet is formed in less than 5 seconds?

4-101. Errors caused by contamination on optical disks occur at the rate of one error every 10^5 bits. Assume the errors follow a Poisson distribution.
(a) What is the mean number of bits until five errors occur?
(b) What is the standard deviation of the number of bits until five errors occur?

(c) The error-correcting code might be ineffective if there are three or more errors within 10^5 bits. What is the probability of this event?

4-102. Calls to the help line of a large computer distributor follow a Possion distribution with a mean of 20 calls per minute.
(a) What is the mean time until the one-hundredth call?
(b) What is the mean time between call numbers 50 and 80?
(c) What is the probability that three or more calls occur within 15 seconds?

4-103. The time between arrivals of customers at an automatic teller machine is an exponential random variable with a mean of 5 minutes.
(a) What is the probability that more than three customers arrive in 10 minutes?
(b) What is the probability that the time until the fifth customer arrives is less than 15 minutes?

4-104. The time between process problems in a manufacturing line is exponentially distributed with a mean of 30 days.
(a) What is the expected time until the fourth problem?
(b) What is the probability that the time until the fourth problem exceeds 120 days?

4-105. Use the properties of the gamma function to evaluate the following:
(a) $\Gamma(6)$ (b) $\Gamma(5/2)$
(c) $\Gamma(9/2)$

4-106. Use integration by parts to show that $\Gamma(r) = (r-1)\Gamma(r-1)$.

4-107. Show that the gamma density function $f(x, \lambda, r)$ integrates to 1.

4-108. Use the result for the gamma distribution to determine the mean and variance of a chi-square distribution with $r = 7/2$.

4-11 WEIBULL DISTRIBUTION

As mentioned previously, the Weibull distribution is often used to model the time until failure of many different physical systems. The parameters in the distribution provide a great deal of flexibility to model systems in which the number of failures increases with time (bearing wear), decreases with time (some semiconductors), or remains constant with time (failures caused by external shocks to the system).

Definition

The random variable X with probability density function

$$f(x) = \frac{\beta}{\delta}\left(\frac{x}{\delta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\delta}\right)^{\beta}\right], \quad \text{for } x > 0 \qquad (4\text{-}22)$$

is a **Weibull random variable** with scale parameter $\delta > 0$ and shape parameter $\beta > 0$.

The flexibility of the Weibull distribution is illustrated by the graphs of selected probability density functions in Fig. 4-27. By inspecting the probability density function, it is seen that when $\beta = 1$, the Weibull distribution is identical to the exponential distribution.

The cumulative distribution function is often used to compute probabilities. The following result can be obtained.

If X has a Weibull distribution with parameters δ and β, then the cumulative distribution function of X is

$$F(x) = 1 - e^{-\left(\frac{x}{\delta}\right)^{\beta}} \qquad (4\text{-}23)$$

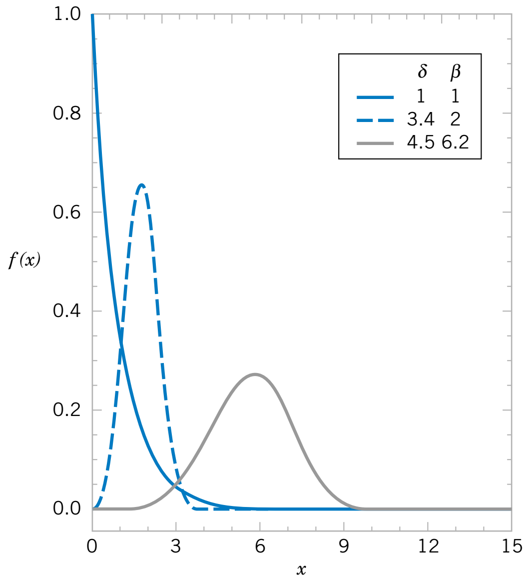


Figure 4-27 Weibull probability density functions for selected values of δ and β.

Also, the following result can be obtained.

If X has a Weibull distribution with parameters δ and β,

$$\mu = E(x) = \delta\Gamma\left(1 + \frac{1}{\beta}\right) \quad \text{and} \quad \sigma^2 = V(x) = \delta^2\Gamma\left(1 + \frac{2}{\beta}\right) - \delta^2\left[\Gamma\left(1 + \frac{1}{\beta}\right)\right]^2 \tag{4-24}$$

EXAMPLE 4-25 The time to failure (in hours) of a bearing in a mechanical shaft is satisfactorily modeled as a Weibull random variable with $\beta = 1/2$, and $\delta = 5000$ hours. Determine the mean time until failure.

From the expression for the mean,

$$E(X) = 5000\Gamma[1 + (1/0.5)] = 5000\Gamma[3] = 5000 \times 2! = 10{,}000 \text{ hours}$$

Determine the probability that a bearing lasts at least 6000 hours. Now

$$P(x > 6000) = 1 - F(6000) = \exp -\left[\left(\frac{6000}{5000}\right)^{1/2}\right] = e^{-1.095} = 0.334$$

Consequently, only 33.4% of all bearings last at least 6000 hours.

EXERCISES FOR SECTION 4-11

4-109. Suppose that X has a Weibull distribution with $\beta = 0.2$ and $\delta = 100$ hours. Determine the mean and variance of X.

4-110. Suppose that X has a Weibull distribution $\beta = 0.2$ and $\delta = 100$ hours. Determine the following:
(a) $P(X < 10{,}000)$ (b) $P(X > 5000)$

4-111. Assume that the life of a roller bearing follows a Weibull distribution with parameters $\beta = 2$ and $\delta = 10{,}000$ hours.
(a) Determine the probability that a bearing lasts at least 8000 hours.
(b) Determine the mean time until failure of a bearing.
(c) If 10 bearings are in use and failures occur independently, what is the probability that all 10 bearings last at least 8000 hours?

4-112. The life (in hours) of a computer processing unit (CPU) is modeled by a Weibull distribution with parameters $\beta = 3$ and $\delta = 900$ hours.
(a) Determine the mean life of the CPU.
(b) Determine the variance of the life of the CPU.
(c) What is the probability that the CPU fails before 500 hours?

4-113. Assume the life of a packaged magnetic disk exposed to corrosive gases has a Weibull distribution with $\beta = 0.5$ and the mean life is 600 hours.
(a) Determine the probability that a packaged disk lasts at least 500 hours.
(b) Determine the probability that a packaged disk fails before 400 hours.

4-114. The life of a recirculating pump follows a Weibull distribution with parameters $\beta = 2$, and $\delta = 700$ hours.
(a) Determine the mean life of a pump.
(b) Determine the variance of the life of a pump.
(c) What is the probability that a pump will last longer than its mean?

4-115. The life (in hours) of a magnetic resonance imaging machine (MRI) is modeled by a Weibull distribution with parameters $\beta = 2$ and $\delta = 500$ hours.
(a) Determine the mean life of the MRI.
(b) Determine the variance of the life of the MRI.
(c) What is the probability that the MRI fails before 250 hours?

4-116. If X is a Weibull random variable with $\beta = 1$, and $\delta = 1000$, what is another name for the distribution of X and what is the mean of X?

4-12 LOGNORMAL DISTRIBUTION

Variables in a system sometimes follow an exponential relationship as $x = exp(w)$. If the exponent is a random variable, say W, $X = \exp(W)$ is a random variable and the distribution of X is of interest. An important special case occurs when W has a normal distribution. In that case, the distribution of X is called a **lognormal distribution.** The name follows from the transformation $\ln(X) = W$. That is, the natural logarithm of X is normally distributed.

Probabilities for X are obtained from the transformation to W, but we need to recognize that the range of X is $(0, \infty)$. Suppose that W is normally distributed with mean θ and variance ω^2; then the cumulative distribution function for X is

$$F(x) = P[X \le x] = P[\exp(W) \le x] = P[W \le \ln(x)]$$
$$= P\left[Z \le \frac{\ln(x) - \theta}{\omega}\right] = \Phi\left[\frac{\ln(x) - \theta}{\omega}\right]$$

for $x > 0$, where Z is a standard normal random variable. Therefore, Appendix Table II can be used to determine the probability. Also, $F(x) = 0$, for $x \le 0$.

The probability density function of X can be obtained from the derivative of $F(x)$. This derivative is applied to the last term in the expression for $F(x)$, the integral of the standard normal density function. Furthermore, from the probability density function, the mean and variance of X can be derived. The details are omitted, but a summary of results follows.

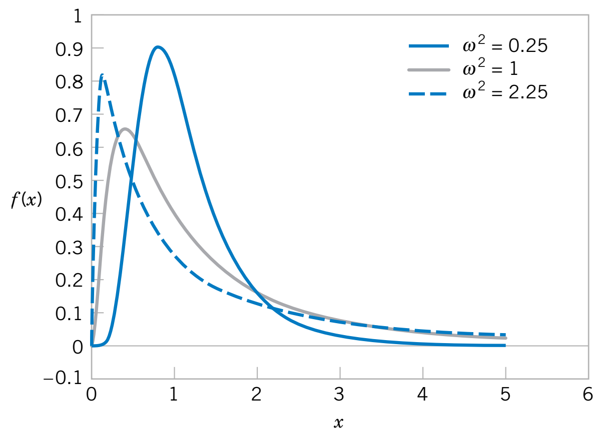


Figure 4-28 Lognormal probability density functions with $\theta = 0$ for selected values of ω^2.

Let W have a normal distribution mean θ and variance ω^2; then $X = \exp(W)$ is a **lognormal random variable** with probability density function

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln x - \theta)^2}{2\omega^2}\right] \qquad 0 < x < \infty$$

The mean and variance of X are

$$E(X) = e^{\theta + \omega^2/2} \quad \text{and} \quad V(X) = e^{2\theta + \omega^2}\left(e^{\omega^2} - 1\right) \tag{4-25}$$

The parameters of a lognormal distribution are θ and ω^2, but care is needed to interpret that these are the mean and variance of the normal random variable W. The mean and variance of X are the functions of these parameters shown in (4-25). Figure 4-28 illustrates lognormal distributions for selected values of the parameters.

The lifetime of a product that degrades over time is often modeled by a lognormal random variable. For example, this is a common distribution for the lifetime of a semiconductor laser. A Weibull distribution can also be used in this type of application, and with an appropriate choice for parameters, it can approximate a selected lognormal distribution. However, a lognormal distribution is derived from a simple exponential function of a normal random variable, so it is easy to understand and easy to evaluate probabilities.

EXAMPLE 4-26

The lifetime of a semiconductor laser has a lognormal distribution with $\theta = 10$ hours and $\omega = 1.5$ hours. What is the probability the lifetime exceeds 10,000 hours?

From the cumulative distribution function for X

$$P(X > 10{,}000) = 1 - P[\exp(W) \le 10{,}000] = 1 - P[W \le \ln(10{,}000)]$$
$$= \Phi\left(\frac{\ln(10{,}000) - 10}{1.5}\right) = 1 - \Phi(-0.52) = 1 - 0.30 = 0.70$$

What lifetime is exceeded by 99% of lasers? The question is to determine x such that $P(X > x) = 0.99$. Therefore,

$$P(X > x) = P[\exp(W) > x] = P[W > \ln(x)] = 1 - \Phi\left(\frac{\ln(x) - 10}{1.5}\right) = 0.99$$

From Appendix Table II, $1 - \Phi(z) = 0.99$ when $z = -2.33$. Therefore,

$$\frac{\ln(x) - 10}{1.5} = -2.33 \quad \text{and} \quad x = \exp(6.505) = 668.48 \text{ hours.}$$

Determine the mean and standard deviation of lifetime. Now,

$$E(X) = e^{\theta + \omega^2/2} = \exp(10 + 1.125) = 67{,}846.3$$
$$V(X) = e^{2\theta + \omega^2}(e^{\omega^2} - 1) = \exp(20 + 2.25)[\exp(2.25) - 1] = 39{,}070{,}059{,}886.6$$

so the standard deviation of X is 197,661.5 hours. Notice that the standard deviation of lifetime is large relative to the mean.

EXERCISES FOR SECTION 4-12

4-117. Suppose that X has a lognormal distribution with parameters $\theta = 5$ and $\omega^2 = 9$. Determine the following:
(a) $P(X < 13{,}300)$
(b) The value for x such that $P(X \le x) = 0.95$
(c) The mean and variance of X

4-118. Suppose that X has a lognormal distribution with parameters $\theta = -2$ and $\omega^2 = 9$. Determine the following:
(a) $P(500 < X < 1000)$
(b) The value for x such that $P(X < x) = 0.1$
(c) The mean and variance of X

4-119. Suppose that X has a lognormal distribution with parameters $\theta = 2$ and $\omega^2 = 4$. Determine the following:
(a) $P(X < 500)$
(b) The conditional probability that $X < 1500$ given that $X > 1000$
(c) What does the difference between the probabilities in parts (a) and (b) imply about lifetimes of lognormal random variables?

4-120. The length of time (in seconds) that a user views a page on a Web site before moving to another page is a lognormal random variable with parameters $\theta = 0.5$ and $\omega^2 = 1$.
(a) What is the probability that a page is viewed for more than 10 seconds?
(b) What is the length of time that 50% of users view the page?
(c) What is the mean and standard deviation of the time until a user moves from the page?

4-121. Suppose that X has a lognormal distribution and that the mean and variance of X are 100 and 85,000, respectively. Determine the parameters θ and ω^2 of the lognormal distribution. (*Hint:* define $x = \exp(\theta)$ and $y = \exp(\omega^2)$ and write two equations in terms of x and y.)

4-122. The lifetime of a semiconductor laser has a lognormal distribution, and it is known that the mean and standard deviation of lifetime are 10,000 and 20,000, respectively.
(a) Calculate the parameters of the lognormal distribution
(b) Determine the probability that a lifetime exceeds 10,000 hours
(c) Determine the lifetime that is exceeded by 90% of lasers

4-123. Derive the probability density function of a lognormal random variable from the derivative of the cumulative distribution function.

Supplemental Exercises

4-124. Suppose that $f(x) = 0.5x - 1$ for $2 < x < 4$. Determine the following:
(a) $P(X < 2.5)$
(b) $P(X > 3)$
(c) $P(2.5 < X < 3.5)$

4-125. Continuation of Exercise 4-124. Determine the cumulative distribution function of the random variable.

4-126. Continuation of Exercise 4-124. Determine the mean and variance of the random variable.

4-127. The time between calls is exponentially distributed with a mean time between calls of 10 minutes.
(a) What is the probability that the time until the first call is less than 5 minutes?
(b) What is the probability that the time until the first call is between 5 and 15 minutes?
(c) Determine the length of an interval of time such that the probability of at least one call in the interval is 0.90.

4-128. Continuation of Exercise 4-127.
(a) If there has not been a call in 10 minutes, what is the probability that the time until the next call is less than 5 minutes?
(b) What is the probability that there are no calls in the intervals from 10:00 to 10:05, from 11:30 to 11:35, and from 2:00 to 2:05?

4-129. Continuation of Exercise 4-127.
(a) What is the probability that the time until the third call is greater than 30 minutes?
(b) What is the mean time until the fifth call?

4-130. The CPU of a personal computer has a lifetime that is exponentially distributed with a mean lifetime of six years. You have owned this CPU for three years. What is the probability that the CPU fails in the next three years?

4-131. Continuation of Exercise 4-130. Assume that your corporation has owned 10 CPUs for three years, and assume that the CPUs fail independently. What is the probability that at least one fails within the next three years?

4-132. Suppose that X has a lognormal distribution with parameters $\theta = 0$ and $\omega^2 = 4$. Determine the following:
(a) $P(10 < X < 50)$
(b) The value for x such that $P(X < x) = 0.05$
(c) The mean and variance of X

4-133. Suppose that X has a lognormal distribution and that the mean and variance of X are 50 and 4000, respectively. Determine the following:
(a) The parameters θ and ω^2 of the lognormal distribution
(b) The probability that X is less than 150

4-134. Asbestos fibers in a dust sample are identified by an electron microscope after sample preparation. Suppose that the number of fibers is a Poisson random variable and the mean number of fibers per squared centimeter of surface dust is 100. A sample of 800 square centimeters of dust is analyzed. Assume a particular grid cell under the microscope represents 1/160,000 of the sample.
(a) What is the probability that at least one fiber is visible in the grid cell?
(b) What is the mean of the number of grid cells that need to be viewed to observe 10 that contain fibers?
(c) What is the standard deviation of the number of grid cells that need to be viewed to observe 10 that contain fibers?

4-135. Without an automated irrigation system, the height of plants two weeks after germination is normally distributed with a mean of 2.5 centimeters and a standard deviation of 0.5 centimeters.
(a) What is the probability that a plant's height is greater than 2.25 centimeters?
(b) What is the probability that a plant's height is between 2.0 and 3.0 centimeters?
(c) What height is exceeded by 90% of the plants?

4-136. Continuation of Exercise 4-135. With an automated irrigation system, a plant grows to a height of 3.5 centimeters two weeks after germination.
(a) What is the probability of obtaining a plant of this height or greater from the distribution of heights in Exercise 4-135.
(b) Do you think the automated irrigation system increases the plant height at two weeks after germination?

4-137. The thickness of a laminated covering for a wood surface is normally distributed with a mean of 5 millimeters and a standard deviation of 0.2 millimeter.
(a) What is the probability that a covering thickness is greater than 5.5 millimeters?
(b) If the specifications require the thickness to be between 4.5 and 5.5 millimeters, what proportion of coverings do not meet specifications?
(c) The covering thickness of 95% of samples is below what value?

4-138. The diameter of the dot produced by a printer is normally distributed with a mean diameter of 0.002 inch. Suppose that the specifications require the dot diameter to be between 0.0014 and 0.0026 inch. If the probability that a dot meets specifications is to be 0.9973, what standard deviation is needed?

4-139. Continuation of Exercise 4-138. Assume that the standard deviation of the size of a dot is 0.0004 inch. If the probability that a dot meets specifications is to be 0.9973, what specifications are needed? Assume that the specifications are to be chosen symmetrically around the mean of 0.002.

4-140. The life of a semiconductor laser at a constant power is normally distributed with a mean of 7000 hours and a standard deviation of 600 hours.
(a) What is the probability that a laser fails before 5,800 hours?
(b) What is the life in hours that 90% of the lasers exceed?

4-141. Continuation of Exercise 4-140. What should the mean life equal in order for 99% of the lasers to exceed 10,000 hours before failure?

4-142. Continuation of Exercise 4-140. A product contains three lasers, and the product fails if any of the lasers fails. Assume the lasers fail independently. What should the mean life equal in order for 99% of the products to exceed 10,000 hours before failure?

4-143. Continuation of Exercise 140. Rework parts (a) and (b). Assume that the lifetime is an exponential random variable with the same mean.

4-144. Continuation of Exercise 4-140. Rework parts (a) and (b). Assume that the lifetime is a lognormal random variable with the same mean and standard deviation.

4-145. A square inch of carpeting contains 50 carpet fibers. The probability of a damaged fiber is 0.0001. Assume the damaged fibers occur independently.

(a) Approximate the probability of one or more damaged fibers in 1 square yard of carpeting.

(b) Approximate the probability of four or more damaged fibers in 1 square yard of carpeting.

4-146. An airline makes 200 reservations for a flight that holds 185 passengers. The probability that a passenger arrives for the flight is 0.9 and the passengers are assumed to be independent.

(a) Approximate the probability that all the passengers that arrive can be seated.

(b) Approximate the probability that there are empty seats.

(c) Approximate the number of reservations that the airline should make so that the probability that everyone who arrives can be seated is 0.95. [*Hint*: Successively try values for the number of reservations.]

MIND-EXPANDING EXERCISES

4-147. The steps in this exercise lead to the probability density function of an Erlang random variable X with parameters λ and r, $f(x) = \lambda^r x^{r-1} e^{-\lambda x}/(r-1)!$, $x > 0$, $r = 1, 2, \ldots$.

(a) Use the Poisson distribution to express $P(X > x)$.

(b) Use the result from part (a) to determine the cumulative distribution function of X.

(c) Differentiate the cumulative distribution function in part (b) and simplify to obtain the probability density function of X.

4-148. A bearing assembly contains 10 bearings. The bearing diameters are assumed to be independent and normally distributed with a mean of 1.5 millimeters and a standard deviation of 0.025 millimeter. What is the probability that the maximum diameter bearing in the assembly exceeds 1.6 millimeters?

4-149. Let the random variable X denote a measurement from a manufactured product. Suppose the target value for the measurement is m. For example, X could denote a dimensional length, and the target might be 10 millimeters. The **quality loss** of the process producing the product is defined to be the expected value of $\$k(X - m)^2$, where k is a constant that relates a deviation from target to a loss measured in dollars.

(a) Suppose X is a continuous random variable with $E(X) = m$ and $V(X) = \sigma^2$. What is the quality loss of the process?

(b) Suppose X is a continuous random variable with $E(X) = \mu$ and $V(X) = \sigma^2$. What is the quality loss of the process?

4-150. The lifetime of an electronic amplifier is modeled as an exponential random variable. If 10% of the amplifiers have a mean of 20,000 hours and the remaining amplifiers have a mean of 50,000 hours, what proportion of the amplifiers fail before 60,000 hours?

4-151. Lack of Memory Property. Show that for an exponential random variable X, $P(X < t_1 + t_2 \mid X > t_1) = P(X < t_2)$

4-152. A process is said to be of **six-sigma quality** if the process mean is at least six standard deviations from the nearest specification. Assume a normally distributed measurement.

(a) If a process mean is centered between the upper and lower specifications at a distance of six standard deviations from each, what is the probability that a product does not meet specifications? Using the result that 0.000001 equals one part per million, express the answer in parts per million.

(b) Because it is difficult to maintain a process mean centered between the specifications, the probability of a product not meeting specifications is often calculated after assuming the process shifts. If the process mean positioned as in part (a) shifts upward by 1.5 standard deviations, what is the probability that a product does not meet specifications? Express the answer in parts per million.

(c) Rework part (a). Assume that the process mean is at a distance of three standard deviations.

(d) Rework part (b). Assume that the process mean is at a distance of three standard deviations and then shifts upward by 1.5 standard deviations.

(e) Compare the results in parts (b) and (d) and comment.

IMPORTANT TERMS AND CONCEPTS

In the E-book, click on any term or concept below to go to that subject.

Chi-squared distribution
Continuous uniform distribution
Cumulative probability distribution function-continuous random variable
Erlang distribution
Exponential distribution
Gamma distribution
Lack of memory property-continuous random variable
Lognormal distribution
Mean-continuous random variable
Mean-function of a continuous random variable
Normal approximation to binomial and Poisson probabilities
Normal distribution
Probability density function
Probability distribution-continuous random variable
Standard deviation-continuous random variable
Standard normal distribution
Standardizing
Variance-continuous random variable
Weibull distribution

CD MATERIAL

Continuity correction

Mean and Variance of the Normal Distribution (CD Only)

In the derivations below, the mean and variance of a normal random variable are shown to be μ and σ^2, respectively. The mean of x is

$$E(X) = \int_{-\infty}^{\infty} x \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dx$$

By making the change of variable $y = (x - \mu)/\sigma$, the integral becomes

$$E(X) = \mu \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy + \sigma \int_{-\infty}^{\infty} y \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

The first integral in the expression above equals 1 because $\frac{e^{-y^2/2}}{\sqrt{2\pi}}$ is a probability density function and the second integral is found to be 0 by either formally making the change of variable $u = -y^2/2$ or noticing the symmetry of the integrand about $y = 0$. Therefore, $E(X) = \mu$.

The variance of X is

$$V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 \frac{e^{-(x-\mu)^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dx$$

By making the change of variable $y = (x - \mu)/\sigma$, the integral becomes

$$V(X) = \sigma^2 \int_{-\infty}^{\infty} y^2 \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$

Upon integrating by parts with $u = y$ and $dv = y \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$, $V(X)$ is found to be σ^2.

4-8 CONTINUITY CORRECTIONS TO IMPROVE THE APPROXIMATION

From Fig. 4-19 it can be seen that a probability such as $P(3 \le X \le 7)$ is better approximated by the area under the normal curve from 2.5 to 7.5. This observation provides a method to improve the approximation of binomial probabilities. Because a continuous normal distribution is used to approximate a discrete binomial distribution, the modification is referred to as a **continuity correction.**

If X is a binomial random variable with parameters n and p, and if $x = 0, 1, 2, \ldots, n$, the **continuity correction** to improve approximations obtained from the normal distribution is

$$P(X \le x) = P(X \le x + 0.5) \cong P\left(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}} \right)$$

and

$$P(x \le X) = P(x - 0.5 \le X) \cong P\left(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} \le Z \right)$$

A way to remember the approximation is to write the probability in terms of $\leq$ or $\geq$ and then add or subtract the 0.5 correction factor to make the probability greater.

EXAMPLE S4-1 Consider the situation in Example 4-20 with $n = 50$ and $p = 0.1$. The probability $P(X \leq 2)$ is better approximated as

$$P(X \leq 2) = P(X \leq 2.5) \approx P\left(Z \leq \frac{2 + 0.5 - 5}{2.12}\right) = P(Z \leq -1.18) = 0.119$$

and this result is closer to the exact probability of 0.112 than the previous result of 0.08.

As another example, $P(8 < X) = P(9 \leq X)$ and this is better approximated as

$$P(9 \leq X) = P(8.5 \leq X) \approx P\left(\frac{9 - 0.5 - 5}{2.12} \leq Z\right) = P(1.65 \leq Z) = 0.05$$

We can even approximate $P(X = 5) = P(5 \leq X \leq 5)$ as

$$P(5 \leq X \leq 5) \approx P\left(\frac{5 - 0.5 - 5}{2.12} \leq Z \leq \frac{5 + 0.5 - 5}{2.12}\right) = P(-0.24 \leq Z \leq 0.24) = 0.19$$

and this compares well with the exact answer of 0.1849.

EXERCISES FOR SECTION 4-8

S4-1. **Continuity correction.** The normal approximation of a binomial probability is sometimes modified by a correction factor of 0.5 that improves the approximation. Suppose that X is binomial with $n = 50$ and $p = 0.1$. Because X is a discrete random variable, $P(X \leq 2) = P(X \leq 2.5)$. However, the normal approximation to $P(X \leq 2)$ can be improved by applying the approximation to $P(X \leq 2.5)$.

(a) Approximate $P(X \leq 2)$ by computing the z-value corresponding to $x = 2.5$.

(b) Approximate $P(X \leq 2)$ by computing the z-value corresponding to $x = 2$.

(c) Compare the results in parts (a) and (b) to the exact value of $P(X \leq 2)$ to evaluate the effectiveness of the continuity correction.

(d) Use the continuity correction to approximate $P(X < 10)$.

S4-2. **Continuity correction.** Suppose that X is binomial with $n = 50$ and $p = 0.1$. Because X is a discrete random variable, $P(X \geq 2) = P(X \geq 1.5)$. However, the normal approximation to $P(X \geq 2)$ can be improved by applying the approximation to $P(X \geq 1.5)$. The continuity correction of 0.5 is either added or subtracted. The easy rule to remember is that the continuity correction is always applied to make the approximating normal probability greatest.

(a) Approximate $P(X \geq 2)$ by computing the z-value corresponding to 1.5.

(b) Approximate $P(X \geq 2)$ by computing the z-value corresponding to 2.

(c) Compare the results in parts (a) and (b) to the exact value of $P(X \geq 2)$ to evaluate the effectiveness of the continuity correction.

(d) Use the continuity correction to approximate $P(X > 6)$.

S4-3. **Continuity correction.** Suppose that X is binomial with $n = 50$ and $p = 0.1$. Because X is a discrete random variable, $P(2 \leq X \leq 5) = P(1.5 \leq X \leq 5.5)$. However, the normal approximation to $P(2 \leq X \leq 5)$ can be improved by applying the approximation to $P(1.5 \leq X \leq 5.5)$.

(a) Approximate $P(2 \leq X \leq 5)$ by computing the z-values corresponding to 1.5 and 5.5.

(b) Approximate $P(2 \leq X \leq 5)$ by computing the z-values corresponding to 2 and 5.

S4-4. **Continuity correction.** Suppose that X is binomial with $n = 50$ and $p = 0.1$. Then, $P(X = 10) = P(10 \leq X \leq 10)$. Using the results for the continuity corrections, we can approximate $P(10 \leq X \leq 10)$ by applying the normal standardization to $P(9.5 \leq X \leq 10.5)$.

(a) Approximate $P(X = 10)$ by computing the z-values corresponding to 9.5 and 10.5.

(b) Approximate $P(X = 5)$.

S4-5. **Continuity correction.** The manufacturing of semiconductor chips produces 2% defective chips. Assume that the chips are independent and that a lot contains 1000 chips.

(a) Use the continuity correction to approximate the probability that 20 to 30 chips in the lot are defective.

(b) Use the continuity correction to approximate the probability that exactly 20 chips are defective.

(c) Determine the number of defective chips, x, such that the normal approximation for the probability of obtaining x defective chips is greatest.