8 Statistical Intervals for a Single Sample

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Construct confidence intervals on the mean of a normal distribution, using either the normal distribution or the t distribution method
2. Construct confidence intervals on the variance and standard deviation of a normal distribution
3. Construct confidence intervals on a population proportion
4. Construct prediction intervals for a future observation
5. Construct a tolerance interval for a normal population

6. **Explain the three types of interval estimates: confidence intervals, prediction intervals, and tolerance intervals**
7. **Use the general method for constructing a confidence interval**

CD MATERIAL

8. **Use the bootstrap technique to construct a confidence interval**

Answers for many odd numbered exercises are at the end of the book. Answers to exercises whose numbers are surrounded by a box can be accessed in the e-Text by clicking on the box. Complete worked solutions to certain exercises are also available in the e-Text. These are indicated in the Answers to Selected Exercises section by a box around the exercise number. Exercises are also available for some of the text sections that appear on CD only. These exercises may be found within the e-Text immediately following the section they accompany.

8-1 INTRODUCTION

In the previous chapter we illustrated how a parameter can be estimated from sample data. However, it is important to understand how good is the estimate obtained. For example, suppose that we estimate the mean viscosity of a chemical product to be $\hat{\mu} = \bar{x} = 1000$. Now because of sampling variability, it is almost never the case that $\mu = \bar{x}$. The point estimate says nothing about how close $\hat{\mu}$ is to μ. Is the process mean likely to be between 900 and 1100? Or is it likely to be between 990 and 1010? The answer to these questions affects our decisions regarding this process. Bounds that represent an interval of plausible values for a parameter are an example of an interval estimate. Surprisingly, it is easy to determine such intervals in many cases, and the same data that provided the point estimate are typically used.

An interval estimate for a population parameter is called a **confidence interval.** We cannot be certain that the interval contains the true, unknown population parameter—we only use a sample from the full population to compute the point estimate and the interval. However, the confidence interval is constructed so that we have high confidence that it does contain the unknown population parameter. Confidence intervals are widely used in engineering and the sciences.

A **tolerance interval** is another important type of interval estimate. For example, the chemical product viscosity data might be assumed to be normally distributed. We might like to calculate limits that bound 95% of the viscosity values. For a normal distribution, we know that 95% of the distribution is in the interval

$$\mu - 1.96\sigma, \mu + 1.96\sigma \tag{8-1}$$

However, this is not a useful tolerance interval because the parameters μ and σ are unknown. Point estimates such as $\bar{x}$ and s can be used in Equation 8-1 for μ and σ. However, we need to account for the potential error in each point estimate to form a tolerance interval for the distribution. The result is an interval of the form

$$\bar{x} - ks, \bar{x} + ks \tag{8-2}$$

where k is an appropriate constant (that is larger than 1.96 to account for the estimation error). As for a confidence interval, it is not certain that Equation 8-2 bounds 95% of the distribution, but the interval is constructed so that we have high confidence that it does. Tolerance intervals are widely used and, as we will subsequently see, they are easy to calculate for normal distributions.

Confidence and tolerance intervals bound unknown elements of a distribution. In this chapter you will learn to appreciate the value of these intervals. A **prediction interval** provides bounds on one (or more) future observations from the population. For example, a prediction interval could be used to bound a single, new measurement of viscosity—another useful interval. With a large sample size, the prediction interval for normally distributed data tends to the tolerance interval in Equation 8-1, but for more modest sample sizes the prediction and tolerance intervals are different.

Keep the purpose of the three types of interval estimates clear:

- A confidence interval bounds population or distribution parameters (such as the mean viscosity).
- A tolerance interval bounds a selected proportion of a distribution.
- A prediction interval bounds future observations from the population or distribution.

8-2 CONFIDENCE INTERVAL ON THE MEAN OF A NORMAL DISTRIBUTION, VARIANCE KNOWN

The basic ideas of a confidence interval (CI) are most easily understood by initially considering a simple situation. Suppose that we have a normal population with unknown mean μ and known variance σ^2. This is a somewhat unrealistic scenario because typically we know the distribution mean before we know the variance. However, in subsequent sections we will present confidence intervals for more general situations.

8-2.1 Development of the Confidence Interval and its Basic Properties

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with unknown mean μ and known variance σ^2. From the results of Chapter 5 we know that the sample mean $\overline{X}$ is normally distributed with mean μ and variance σ^2/n. We may **standardize** $\overline{X}$ by subtracting the mean and dividing by the standard deviation, which results in the variable

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{8-3}$$

Now Z has a standard normal distribution.

A **confidence interval** estimate for μ is an interval of the form $l \leq \mu \leq u$, where the end-points l and u are computed from the sample data. Because different samples will produce different values of l and u, these end-points are values of random variables L and U, respectively. Suppose that we can determine values of L and U such that the following probability statement is true:

$$P\{L \leq \mu \leq U\} = 1 - \alpha \tag{8-4}$$

where $0 \leq \alpha \leq 1$. There is a probability of $1 - \alpha$ of selecting a sample for which the CI will contain the true value of μ. Once we have selected the sample, so that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and computed l and u, the resulting **confidence interval** for μ is

$$l \leq \mu \leq u \tag{8-5}$$

The end-points or bounds l and u are called the **lower-** and **upper-confidence limits,** respectively, and $1 - \alpha$ is called the **confidence coefficient.**

In our problem situation, because $Z = (\overline{X} - \mu)/(\sigma/\sqrt{n})$ has a standard normal distribution, we may write

$$P\left\{-z_{\alpha/2} \le \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \le z_{\alpha/2}\right\} = 1 - \alpha$$

Now manipulate the quantities inside the brackets by (1) multiplying through by $\sigma/\sqrt{n}$, (2) subtracting $\overline{X}$ from each term, and (3) multiplying through by -1. This results in

$$P\left\{\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\} = 1 - \alpha \qquad (8\text{-}6)$$

From consideration of Equation 8-4, the lower and upper limits of the inequalities in Equation 8-6 are the lower- and upper-confidence limits L and U, respectively. This leads to the following definition.

Definition

If $\bar{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1 - \alpha)\%$ CI on μ is given by

$$\bar{x} - z_{\alpha/2}\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{\alpha/2}\sigma/\sqrt{n} \qquad (8\text{-}7)$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the standard normal distribution.

EXAMPLE 8-1

ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at 60ºC are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, and 64.3. Assume that impact energy is normally distributed with $\sigma = 1J$. We want to find a 95% CI for μ, the mean impact energy. The required quantities are $z_{\alpha/2} = z_{0.025} = 1.96$, $n = 10$, $\sigma = 1$, and $\bar{x} = 64.46$. The resulting 95% CI is found from Equation 8-7 as follows:

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$64.46 - 1.96\frac{1}{\sqrt{10}} \le \mu \le 64.46 + 1.96\frac{1}{\sqrt{10}}$$

$$63.84 \le \mu \le 65.08$$

That is, based on the sample data, a range of highly plausible vaules for mean impact energy for A238 steel at 60°C is $63.84J \le \mu \le 65.08J$.

Interpreting a Confidence Interval

How does one interpret a confidence interval? In the impact energy estimation problem in Example 8-1 the 95% CI is $63.84 \le \mu \le 65.08$, so it is tempting to conclude that μ is within this interval with probability 0.95. However, with a little reflection, it's easy to see that this cannot be correct; the true value of μ is unknown and the statement $63.84 \leq \mu \leq 65.08$ is either correct (true with probability 1) or incorrect (false with probability 1). The correct interpretation lies in the realization that a CI is **a random interval** because in the probability statement defining the end-points of the interval (Equation 8-4), L and U are random variables. Consequently, the correct interpretation of a $100(1 - \alpha)\%$ CI depends on the relative frequency view of probability. Specifically, if an infinite number of random samples are collected and a $100(1 - \alpha)\%$ confidence interval for μ is computed from each sample, $100(1 - \alpha)\%$ of these intervals will contain the true value of μ.

The situation is illustrated in Fig. 8-1, which shows several $100(1 - \alpha)\%$ confidence intervals for the mean μ of a normal distribution. The dots at the center of the intervals indicate the point estimate of μ (that is, $\bar{x}$). Notice that one of the intervals fails to contain the true value of μ. If this were a 95% confidence interval, in the long run only 5% of the intervals would fail to contain μ.

Now in practice, we obtain only one random sample and calculate one confidence interval. Since this interval either will or will not contain the true value of μ, it is not reasonable to attach a probability level to this specific event. The appropriate statement is the observed interval $[l, u]$ brackets the true value of μ with **confidence** $100(1 - \alpha)$. This statement has a frequency interpretation; that is, we don't know if the statement is true for this specific sample, but the **method** used to obtain the interval $[l, u]$ yields correct statements $100(1 - \alpha)\%$ of the time.

Confidence Level and Precision of Estimation

Notice in Example 8-1 that our choice of the 95% level of confidence was essentially arbitrary. What would have happened if we had chosen a higher level of confidence, say, 99%? In fact, doesn't it seem reasonable that we would want the higher level of confidence? At $\alpha = 0.01$, we find $z_{\alpha/2} = z_{0.01/2} = z_{0.005} = 2.58$, while for $\alpha = 0.05$, $z_{0.025} = 1.96$. Thus, the **length** of the 95% confidence interval is

$$2(1.96\sigma/\sqrt{n}) = 3.92\sigma/\sqrt{n}$$

whereas the length of the 99% CI is

$$2(2.58\sigma/\sqrt{n}) = 5.16\sigma/\sqrt{n}$$

Thus, the 99% CI is longer than the 95% CI. This is why we have a higher level of confidence in the 99% confidence interval. Generally, for a fixed sample size n and standard deviation σ, the higher the confidence level, the longer the resulting CI.

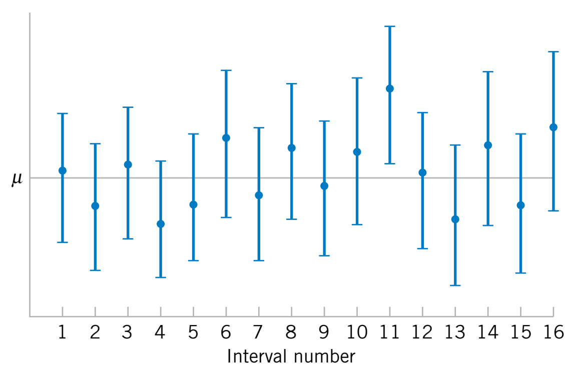


Figure 8-1 Repeated construction of a confidence interval for μ.

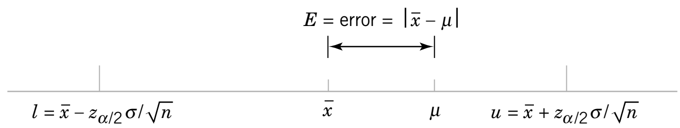


Figure 8-2 Error in estimating μ with $\bar{x}$.

The length of a confidence interval is a measure of the **precision** of estimation. From the preceeding discussion, we see that precision is inversely related to the confidence level. It is desirable to obtain a confidence interval that is short enough for decision-making purposes and that also has adequate confidence. One way to achieve this is by choosing the sample size n to be large enough to give a CI of specified length or precision with prescribed confidence.

8-2.2 Choice of Sample Size

The precision of the confidence interval in Equation 8-7 is $2z_{\alpha/2}\sigma/\sqrt{n}$. This means that in using $\bar{x}$ to estimate μ, the error $E = |\bar{x} - \mu|$ is less than or equal to $z_{\alpha/2}\sigma/\sqrt{n}$ with confidence $100(1 - \alpha)$. This is shown graphically in Fig. 8-2. In situations where the sample size can be controlled, we can choose n so that we are $100(1 - \alpha)$ percent confident that the error in estimating μ is less than a specified bound on the error E. The appropriate sample size is found by choosing n such that $z_{\alpha/2}\sigma/\sqrt{n} = E$. Solving this equation gives the following formula for n.

Definition

If $\bar{x}$ is used as an estimate of μ, we can be $100(1 - \alpha)\%$ confident that the error $|\bar{x} - \mu|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \quad (8\text{-}8)$$

If the right-hand side of Equation 8-8 is not an integer, it must be rounded up. This will ensure that the level of confidence does not fall below $100(1 - \alpha)\%$. Notice that $2E$ is the length of the resulting confidence interval.

EXAMPLE 8-2

To illustrate the use of this procedure, consider the CVN test described in Example 8-1, and suppose that we wanted to determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut at 60°C has a length of at most $1.0J$. Since the bound on error in estimation E is one-half of the length of the CI, to determine n we use Equation 8-8 with $E = 0.5$, $\sigma = 1$, and $z_{\alpha/2} = 0.025$. The required sample size is 16

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left[\frac{(1.96)1}{0.5}\right]^2 = 15.37$$

and because n must be an integer, the required sample size is $n = 16$.

Notice the general relationship between sample size, desired length of the confidence interval $2E$, confidence level $100(1 - \alpha)$, and standard deviation σ:

- As the desired length of the interval $2E$ decreases, the required sample size n increases for a fixed value of σ and specified confidence.

- As σ increases, the required sample size n increases for a fixed desired length $2E$ and specified confidence.
- As the level of confidence increases, the required sample size n increases for fixed desired length $2E$ and standard deviation σ.

8-2.3 One-Sided Confidence Bounds

The confidence interval in Equation 8-7 gives both a lower confidence bound and an upper confidence bound for μ. Thus it provides a two-sided CI. It is also possible to obtain one-sided confidence bounds for μ by setting either $l = -\infty$ or $u = \infty$ and replacing $z_{\alpha/2}$ by z_α.

Definition

A $100(1 - \alpha)\%$ **upper-confidence bound** for μ is

$$\mu \leq u = \bar{x} + z_\alpha \sigma/\sqrt{n} \tag{8-9}$$

and a $100(1 - \alpha)\%$ **lower-confidence bound** for μ is

$$\bar{x} - z_\alpha \sigma/\sqrt{n} = l \leq \mu \tag{8-10}$$

8-2.4 General Method to Derive a Confidence Interval

It is easy to give a general method for finding a confidence interval for an unknown parameter θ. Let $X_1, X_2, \ldots, X_n$ be a random sample of n observations. Suppose we can find a statistic $g(X_1, X_2, \ldots, X_n; \theta)$ with the following properties:

1. $g(X_1, X_2, \ldots, X_n; \theta)$ depends on both the sample and θ.
2. The probability distribution of $g(X_1, X_2, \ldots, X_n; \theta)$ does not depend on θ or any other unknown parameter.

In the case considered in this section, the parameter $\theta = \mu$. The random variable $g(X_1, X_2, \ldots, X_n; \mu) = (\bar{X} - \mu)/(\sigma/\sqrt{n})$ and satisfies both conditions above; it depends on the sample and on μ, and it has a standard normal distribution since σ is known. Now one must find constants C_L and C_U so that

$$P[C_L \leq g(X_1, X_2, \ldots, X_n; \theta) \leq C_U] = 1 - \alpha \tag{8-11}$$

Because of property 2, C_L and C_U do not depend on θ. In our example, $C_L = -z_{\alpha/2}$ and $C_U = z_{\alpha/2}$. Finally, you must manipulate the inequalities in the probability statement so that

$$P[L(X_1, X_2, \ldots, X_n) \leq \theta \leq U(X_1, X_2, \ldots, X_n)] = 1 - \alpha \tag{8-12}$$

This gives $L(X_1, X_2, \ldots, X_n)$ and $U(X_1, X_2, \ldots, X_n)$ as the lower and upper confidence limits defining the $100(1 - \alpha)\%$ confidence interval for θ. The quantity $g(X_1, X_2, \ldots, X_n; \theta)$ is often called a "pivotal quantity" because we pivot on this quantity in Equation 8-11 to produce Equation 8-12. In our example, we manipulated the pivotal quantity $(\bar{X} - \mu)/(\sigma/\sqrt{n})$ to obtain $L(X_1, X_2, \ldots, X_n) = \bar{X} - z_{\alpha/2}\sigma/\sqrt{n}$ and $U(X_1, X_2, \ldots, X_n) = \bar{X} + z_{\alpha/2}\sigma/\sqrt{n}$.

8-2.5 A Large-Sample Confidence Interval for μ

We have assumed that the population distribution is normal with unknown mean and known standard deviation σ. We now present a **large-sample CI** and μ that does not require these assumptions. Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with unknown mean μ and variance σ^2. Now if the sample size n is large, the central limit theorem implies that $\overline{X}$ has approximately a normal distribution with mean μ and variance σ^2/n. Therefore $Z = (\overline{X} - \mu)/(\sigma/\sqrt{n})$ has approximately a standard normal distribution. This ratio could be used as a pivotal quantity and manipulated as in Section 8-2.1 to produce an approximate CI for μ. However, the standard deviation σ is unknown. It turns out that when n is large, replacing σ by the sample standard deviation S has little effect on the distribution of Z. This leads to the following useful result.

Definition

When n is large, the quantity

$$\frac{\overline{X} - \mu}{S/\sqrt{n}}$$

has an approximate standard normal distribution. Consequently,

$$\bar{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{\alpha/2}\frac{s}{\sqrt{n}} \tag{8-13}$$

is a **large sample confidence interval** for μ, with confidence level of approximately $100(1 - \alpha)\%$.

Equation 8-13 holds regardless of the shape of the population distribution. Generally n should be at least 40 to use this result reliably. The central limit theorem generally holds for $n \geq 30$, but the larger sample size is recommended here because replacing σ by S in Z results in additional variability.

EXAMPLE 8-3 An article in the 1993 volume of the *Transactions of the American Fisheries Society* reports the results of a study to investigate the mercury contamination in largemouth bass. A sample of fish was selected from 53 Florida lakes and mercury concentration in the muscle tissue was measured (ppm). The mercury concentration values are

1.230	0.490	0.490	1.080	0.590	0.280	0.180	0.100	0.940
1.330	0.190	1.160	0.980	0.340	0.340	0.190	0.210	0.400
0.040	0.830	0.050	0.630	0.340	0.750	0.040	0.860	0.430
0.044	0.810	0.150	0.560	0.840	0.870	0.490	0.520	0.250
1.200	0.710	0.190	0.410	0.500	0.560	1.100	0.650	0.270
0.270	0.500	0.770	0.730	0.340	0.170	0.160	0.270	

The summary statistics from Minitab are displayed below:

Descriptive Statistics: Concentration

Variable	N	Mean	Median	TrMean	StDev	SE Mean
Concentration	53	0.5250	0.4900	0.5094	0.3486	0.0479

Variable	Minimum	Maximum	Q1	Q3
Concentration	0.0400	1.3300	0.2300	0.7900

Figure 8-3(a) and (b) presents the histogram and normal probability plot of the mercury concentration data. Both plots indicate that the distribution of mercury concentration is not normal and is positively skewed. We want to find an approximate 95% CI on μ. Because $n > 40$, the assumption of normality is not necessary to use Equation 8-13. The required quantities are $n = 53$, $\bar{x} = 0.5250$, $s = 0.3486$, and $z_{0.025} = 1.96$. The approximate 95% CI on μ is

$$\bar{x} - z_{0.025}\frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + z_{0.025}\frac{s}{\sqrt{n}}$$

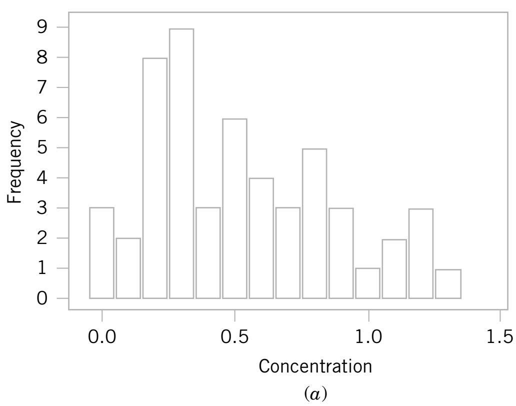
$$0.5250 - 1.96\frac{0.3486}{\sqrt{53}} \leq \mu \leq 0.5250 + 1.96\frac{0.3486}{\sqrt{53}}$$

$$0.4311 \leq \mu \leq 0.6189$$

This interval is fairly wide because there is a lot of variability in the mercury concentration measurements.

A General Large Sample Confidence Interval

The large-sample confidence interval for μ in Equation 8-13 is a special case of a more general result. Suppose that θ is a parameter of a probability distribution and let $\hat{\Theta}$ be an estimator of θ. If $\hat{\Theta}$ (1) has an approximate normal distribution, (2) is approximately unbiased



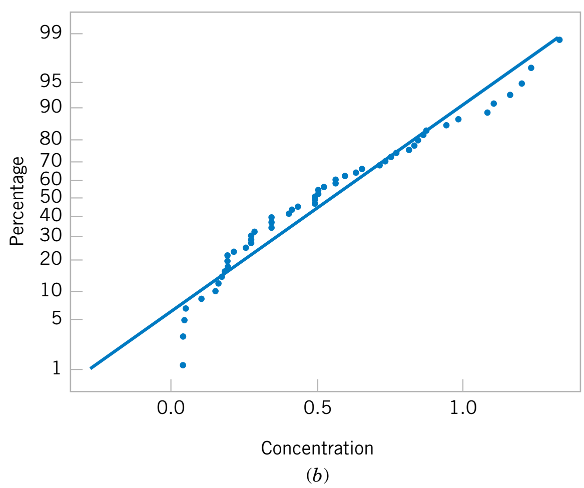


Figure 8-3 Mercury concentration in largemouth bass (a) Histogram. (b) Normal probability plot.

for θ, and (3) has standard deviation $\sigma_{\hat{\Theta}}$ that can be estimated from the sample data, then the quantity $(\hat{\Theta} - \theta)/\sigma_{\hat{\Theta}}$ has an approximate standard normal distribution. Then a **large-sample approximate CI for θ** is given by

$$\hat{\theta} - z_{\alpha/2}\sigma_{\hat{\Theta}} \leq \theta \leq \hat{\theta} + z_{\alpha/2}\sigma_{\hat{\Theta}} \qquad (8\text{-}14)$$

Maximum likelihood estimators usually satisfy the three conditions listed above, so Equation 8-14 is often used when $\hat{\Theta}$ is the maximum likelihood estimator of θ. Finally, note that Equation 8-14 can be used even when $\sigma_{\hat{\Theta}}$ is a function of other unknown parameters (or of θ). Essentially, all one does is to use the sample data to compute estimates of the unknown parameters and substitute those estimates into the expression for $\sigma_{\hat{\Theta}}$.

8-2.6 Bootstrap Confidence Intervals (CD Only)

EXERCISES FOR SECTION 8-2

8-1. For a normal population with known variance σ^2, answer the following questions:
(a) What is the confidence level for the interval $\bar{x} - 2.14\sigma/\sqrt{n} \leq \mu \leq \bar{x} + 2.14\sigma/\sqrt{n}$?
(b) What is the confidence level for the interval $\bar{x} - 2.49\sigma/\sqrt{n} \leq \mu \leq \bar{x} + 2.49\sigma/\sqrt{n}$?
(c) What is the confidence level for the interval $\bar{x} - 1.85\sigma/\sqrt{n} \leq \mu \leq \bar{x} + 1.85\sigma/\sqrt{n}$?

8-2. For a normal population with known variance σ^2:
(a) What value of $z_{\alpha/2}$ in Equation 8-7 gives 98% confidence?
(b) What value of $z_{\alpha/2}$ in Equation 8-7 gives 80% confidence?
(c) What value of $z_{\alpha/2}$ in Equation 8-7 gives 75% confidence?

8-3. Consider the one-sided confidence interval expressions, Equations 8-9 and 8-10.
(a) What value of z_α would result in a 90% CI?
(b) What value of z_α would result in a 95% CI?
(c) What value of z_α would result in a 99% CI?

8-4. A confidence interval estimate is desired for the gain in a circuit on a semiconductor device. Assume that gain is normally distributed with standard deviation $\sigma = 20$.
(a) Find a 95% CI for μ when $n = 10$ and $\bar{x} = 1000$.
(b) Find a 95% CI for μ when $n = 25$ and $\bar{x} = 1000$.
(c) Find a 99% CI for μ when $n = 10$ and $\bar{x} = 1000$.
(d) Find a 99% CI for μ when $n = 25$ and $\bar{x} = 1000$.

8-5. Consider the gain estimation problem in Exercise 8-4. How large must n be if the length of the 95% CI is to be 40?

8-6. Following are two confidence interval estimates of the mean μ of the cycles to failure of an automotive door latch mechanism (the test was conducted at an elevated stress level to accelerate the failure).

$$3124.9 \leq \mu \leq 3215.7 \qquad 3110.5 \leq \mu \leq 3230.1$$

(a) What is the value of the sample mean cycles to failure?
(b) The confidence level for one of these CIs is 95% and the confidence level for the other is 99%. Both CIs are calculated from the same sample data. Which is the 95% CI? Explain why.

8-7. $n = 100$ random samples of water from a fresh water lake were taken and the calcium concentration (milligrams per liter) measured. A 95% CI on the mean calcium concentration is $0.49 \leq \mu \leq 0.82$.
(a) Would a 99% CI calculated from the same sample data been longer or shorter?
(b) Consider the following statement: There is a 95% chance that μ is between 0.49 and 0.82. Is this statement correct? Explain your answer.
(c) Consider the following statement: If $n = 100$ random samples of water from the lake were taken and the 95% CI on μ computed, and this process was repeated 1000 times, 950 of the CIs will contain the true value of μ. Is this statement correct? Explain your answer.

8-8. The breaking strength of yarn used in manufacturing drapery material is required to be at least 100 psi. Past experience has indicated that breaking strength is normally distributed and that $\sigma = 2$ psi. A random sample of nine specimens is tested, and the average breaking strength is found to be 98 psi. Find a 95% two-sided confidence interval on the true mean breaking strength.

8-9. The yield of a chemical process is being studied. From previous experience yield is known to be normally distributed and $\sigma = 3$. The past five days of plant operation have resulted in the following percent yields: 91.6, 88.75, 90.8, 89.95, and 91.3. Find a 95% two-sided confidence interval on the true mean yield.

8-10. The diameter of holes for cable harness is known to have a normal distribution with $\sigma = 0.01$ inch. A random sample of size 10 yields an average diameter of 1.5045 inch. Find a 99% two-sided confidence interval on the mean hole diameter.

8-11. A manufacturer produces piston rings for an automobile engine. It is known that ring diameter is normally distributed with $\sigma = 0.001$ millimeters. A random sample of 15 rings has a mean diameter of $\bar{x} = 74.036$ millimeters.

(a) Construct a 99% two-sided confidence interval on the mean piston ring diameter.

(b) Construct a 95% lower-confidence bound on the mean piston ring diameter.

8-12. The life in hours of a 75-watt light bulb is known to be normally distributed with $\sigma = 25$ hours. A random sample of 20 bulbs has a mean life of $\bar{x} = 1014$ hours.

(a) Construct a 95% two-sided confidence interval on the mean life.

(b) Construct a 95% lower-confidence bound on the mean life.

8-13. A civil engineer is analyzing the compressive strength of concrete. Compressive strength is normally distributed with $\sigma^2 = 1000(\text{psi})^2$. A random sample of 12 specimens has a mean compressive strength of $\bar{x} = 3250$ psi.

(a) Construct a 95% two-sided confidence interval on mean compressive strength.

(b) Construct a 99% two-sided confidence interval on mean compressive strength. Compare the width of this confidence interval with the width of the one found in part (a).

8-14. Suppose that in Exercise 8-12 we wanted to be 95% confident that the error in estimating the mean life is less than five hours. What sample size should be used?

8-15. Suppose that in Exercise 8-12 we wanted the total width of the two-sided confidence interval on mean life to be six hours at 95% confidence. What sample size should be used?

8-16. Suppose that in Exercise 8-13 it is desired to estimate the compressive strength with an error that is less than 15 psi at 99% confidence. What sample size is required?

8-17. By how much must the sample size n be increased if the length of the CI on μ in Equation 8-7 is to be halved?

8-18. If the sample size n is doubled, by how much is the length of the CI on μ in Equation 8-7 reduced? What happens to the length of the interval if the sample size is increased by a factor of four?

8-3 CONFIDENCE INTERVAL ON THE MEAN OF A NORMAL DISTRIBUTION, VARIANCE UNKNOWN

When we are constructing confidence intervals on the mean μ of a normal population when σ^2 is known, we can use the procedure in Section 8-2.1. This CI is also approximately valid (because of the central limit theorem) regardless of whether or not the underlying population is normal, so long as n is reasonably large ($n \geq 40$, say). As noted in Section 8-2.5, we can even handle the case of unknown variance for the large-sample-size situation. However, when the sample is small and σ^2 is unknown, we must make an assumption about the form of the underlying distribution to obtain a valid CI procedure. A reasonable assumption in many cases is that the underlying distribution is **normal.**

Many populations encountered in practice are well approximated by the normal distribution, so this assumption will lead to confidence interval procedures of wide applicability. In fact, moderate departure from normality will have little effect on validity. When the assumption is unreasonable, an alternate is to use the nonparametric procedures in Chapter 15 that are valid for any underlying distribution.

Suppose that the population of interest has a normal distribution with unknown mean μ and unknown variance σ^2. Assume that a random sample of size n, say $X_1, X_2, \ldots, X_n$, is available, and let $\bar{X}$ and S^2 be the sample mean and variance, respectively.

We wish to construct a two-sided CI on μ. If the variance σ^2 is known, we know that $Z = (\bar{X} - \mu)/(\sigma/\sqrt{n})$ has a standard normal distribution. When σ^2 is unknown, a logical procedure is to replace σ with the sample standard deviation S. The random variable Z now becomes $T = (\bar{X} - \mu)/(S/\sqrt{n})$. A logical question is what effect does replacing σ by S have on the distribution of the random variable T? If n is large, the answer to this question is "very little," and we can proceed to use the confidence interval based on the normal distribution from

Section 8-2.5. However, n is usually small in most engineering problems, and in this situation a different distribution must be employed to construct the CI.

8-3.1 The t Distribution

Definition

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \tag{8-15}$$

has a t distribution with $n - 1$ degrees of freedom.

The t probability density function is

$$f(x) = \frac{\Gamma[(k+1)/2]}{\sqrt{\pi k}\,\Gamma(k/2)} \cdot \frac{1}{[(x^2/k) + 1]^{(k+1)/2}} \quad -\infty < x < \infty \tag{8-16}$$

where k is the number of degrees of freedom. The mean and variance of the t distribution are zero and $k/(k-2)$ (for $k > 2$), respectively.

Several t distributions are shown in Fig. 8-4. The general appearance of the t distribution is similar to the standard normal distribution in that both distributions are symmetric and unimodal, and the maximum ordinate value is reached when the mean $\mu = 0$. However, the t distribution has heavier tails than the normal; that is, it has more probability in the tails than the normal distribution. As the number of degrees of freedom $k \rightarrow \infty$, the limiting form of the t distribution is the standard normal distribution. Generally, the number of degrees of freedom for t are the number of degrees of freedom associated with the estimated standard deviation.

Appendix Table IV provides **percentage points** of the t distribution. We will let $t_{\alpha,k}$ be the value of the random variable T with k degrees of freedom above which we find an area (or probability) α. Thus, $t_{\alpha,k}$ is an upper-tail 100α percentage point of the t distribution with k degrees of freedom. This percentage point is shown in Fig. 8-5. In the Appendix Table IV the α values are the column headings, and the degrees of freedom are listed in the left column. To

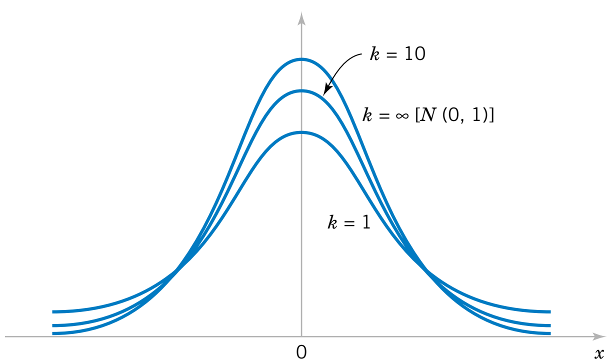


Figure 8-4 Probability density functions of several t distributions.

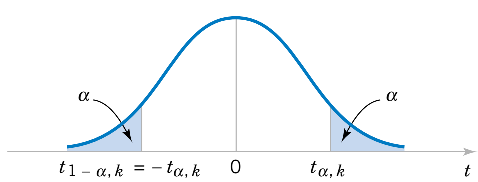


Figure 8-5 Percentage points of the t distribution.

illustrate the use of the table, note that the t-value with 10 degrees of freedom having an area of 0.05 to the right is $t_{0.05,10} = 1.812$. That is,

$$P(T_{10} > t_{0.05,10}) = P(T_{10} > 1.812) = 0.05$$

Since the t distribution is symmetric about zero, we have $t_{1-\alpha} = -t_\alpha$; that is, the t-value having an area of $1 - \alpha$ to the right (and therefore an area of α to the left) is equal to the negative of the t-value that has area α in the right tail of the distribution. Therefore, $t_{0.95,10} = -t_{0.05,10} = -1.812$. Finally, because t_∞ is the standard normal distribution, the familiar z_α values appear in the last row of Appendix Table IV.

8-3.2 Development of the *t* Distribution (CD Only)

8-3.3 The *t* Confidence Interval on μ

It is easy to find a $100(1 - \alpha)$ percent confidence interval on the mean of a normal distribution with unknown variance by proceeding essentially as we did in Section 8-2.1. We know that the distribution of $T = (\overline{X} - \mu)/(S/\sqrt{n})$ is t with $n - 1$ degrees of freedom. Letting $t_{\alpha/2,n-1}$ be the upper $100\alpha/2$ percentage point of the t distribution with $n - 1$ degrees of freedom, we may write:

$$P(-t_{\alpha/2,n-1} \le T \le t_{\alpha/2,n-1}) = 1 - \alpha$$

or

$$P\left(-t_{\alpha/2,n-1} \le \frac{\overline{X} - \mu}{S/\sqrt{n}} \le t_{\alpha/2,n-1}\right) = 1 - \alpha$$

Rearranging this last equation yields

$$P(\overline{X} - t_{\alpha/2,n-1}S/\sqrt{n} \le \mu \le \overline{X} + t_{\alpha/2,n-1}S/\sqrt{n}) = 1 - \alpha \qquad (8\text{-}17)$$

This leads to the following definition of the $100(1 - \alpha)$ percent two-sided confidence interval on μ.

Definition

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, a **$100(1 - \alpha)$ percent confidence interval on μ** is given by

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n} \qquad (8\text{-}18)$$

where $t_{\alpha/2,n-1}$ is the upper $100\alpha/2$ percentage point of the t distribution with $n - 1$ degrees of freedom.

One-sided confidence bounds on the mean of a normal distribution are also of interest and are easy to find. Simply use only the appropriate lower or upper confidence limit from Equation 8-18 and replace $t_{\alpha/2,n-1}$ by $t_{\alpha,n-1}$.

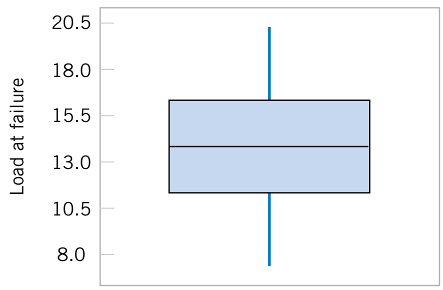


Figure 8-6 Box and whisker plot for the load at failure data in Example 8-4.

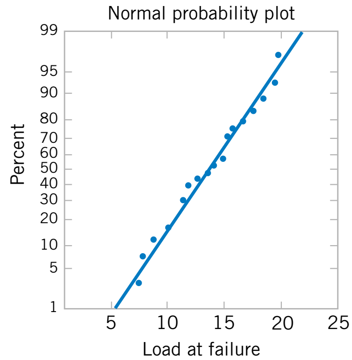


Figure 8-7 Normal probability plot of the load at failure data from Example 8-4.

EXAMPLE 8-4

An article in the journal *Materials Engineering* (1989, Vol. II, No. 4, pp. 275–281) describes the results of tensile adhesion tests on 22 U-700 alloy specimens. The load at specimen failure is as follows (in megapascals):

19.8	10.1	14.9	7.5	15.4	15.4
15.4	18.5	7.9	12.7	11.9	11.4
11.4	14.1	17.6	16.7	15.8	
19.5	8.8	13.6	11.9	11.4	

The sample mean is $\bar{x} = 13.71$, and the sample standard deviation is $s = 3.55$. Figures 8-6 and 8-7 show a box plot and a normal probability plot of the tensile adhesion test data, respectively. These displays provide good support for the assumption that the population is normally distributed. We want to find a 95% CI on μ. Since $n = 22$, we have $n - 1 = 21$ degrees of freedom for t, so $t_{0.025,21} = 2.080$. The resulting CI is

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \leq \mu \leq \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n}$$
$$13.71 - 2.080(3.55)/\sqrt{22} \leq \mu \leq 13.71 + 2.080(3.55)/\sqrt{22}$$
$$13.71 - 1.57 \leq \mu \leq 13.71 + 1.57$$
$$12.14 \leq \mu \leq 15.28$$

The CI is fairly wide because there is a lot of variability in the tensile adhesion test measurements.

It is not as easy to select a sample size n to obtain a specified length (or precision of estimation) for this CI as it was in the known-σ case because the length of the interval involves s (which is unknown before the data are collected), n, and $t_{\alpha/2,n-1}$. Note that the t-percentile depends on the sample size n. Consequently, an appropriate n can only be obtained through trial and error. The results of this will, of course, also depend on the reliability of our prior "guess" for σ.

EXERCISES FOR SECTION 8-3

8-19. Find the values of the following percentiles: $t_{0.025,15}$, $t_{0.05,10}$, $t_{0.10,20}$, $t_{0.005,25}$, and $t_{0.001,30}$.

8-20. Determine the t-percentile that is required to construct each of the following two-sided confidence intervals:
(a) Confidence level = 95%, degrees of freedom = 12
(b) Confidence level = 95%, degrees of freedom = 24
(c) Confidence level = 99%, degrees of freedom = 13
(d) Confidence level = 99.9%, degrees of freedom = 15

8-21. Determine the t-percentile that is required to construct each of the following one-sided confidence intervals:
(a) Confidence level = 95%, degrees of freedom = 14
(b) Confidence level = 99%, degrees of freedom = 19
(c) Confidence level = 99.9%, degrees of freedom = 24

8-22. A research engineer for a tire manufacturer is investigating tire life for a new rubber compound and has built 16 tires and tested them to end-of-life in a road test. The sample mean and standard deviation are 60,139.7 and 3645.94 kilometers. Find a 95% confidence interval on mean tire life.

8-23. An Izod impact test was performed on 20 specimens of PVC pipe. The sample mean is $\bar{x} = 1.25$ and the sample standard deviation is $s = 0.25$. Find a 99% lower confidence bound on Izod impact strength.

8-24. The brightness of a television picture tube can be evaluated by measuring the amount of current required to achieve a particular brightness level. A sample of 10 tubes results in $\bar{x} = 317.2$ and $s = 15.7$. Find (in microamps) a 99% confidence interval on mean current required. State any necessary assumptions about the underlying distribution of the data.

8-25. A particular brand of diet margarine was analyzed to determine the level of polyunsaturated fatty acid (in percentages). A sample of six packages resulted in the following data: 16.8, 17.2, 17.4, 16.9, 16.5, 17.1.

(a) Is there evidence to support the assumption that the level of polyunsaturated fatty acid is normally distributed?
(b) Find a 99% confidence interval on the mean μ. Provide a practical interpretation of this interval.

8-26. The compressive strength of concrete is being tested by a civil engineer. He tests 12 specimens and obtains the following data.

2216	2237	2249	2204
2225	2301	2281	2263
2318	2255	2275	2295

(a) Is there evidence to support the assumption that compressive strength is normally distributed? Does this data set support your point of view? Include a graphical display in your answer.
(b) Construct a 95% two-sided confidence interval on the mean strength.
(c) Construct a 95% lower-confidence bound on the mean strength.

8-27. A machine produces metal rods used in an automobile suspension system. A random sample of 15 rods is selected, and the diameter is measured. The resulting data (in millimeters) are as follows:

8.24	8.25	8.20	8.23	8.24
8.21	8.26	8.26	8.20	8.25
8.23	8.23	8.19	8.28	8.24

(a) Check the assumption of normality for rod diameter.
(b) Find a 95% two-sided confidence interval on mean rod diameter.

8-28. Rework Exercise 8-27 to compute a 95% lower confidence bound on rod diameter. Compare this bound with the lower limit of the two-sided confidence limit from Exercise 8-27. Discuss why they are different.

8-29. The wall thickness of 25 glass 2-liter bottles was measured by a quality-control engineer. The sample mean was $\bar{x} = 4.05$ millimeters, and the sample standard deviation was $s = 0.08$ millimeter. Find a 95% lower confidence bound for mean wall thickness. Interpret the interval you have obtained.

8-30. An article in *Nuclear Engineering International* (February 1988, p. 33) describes several characteristics of fuel rods used in a reactor owned by an electric utility in Norway. Measurements on the percentage of enrichment of 12 rods were reported as follows:

2.94	3.00	2.90	2.75	3.00	2.95
2.90	2.75	2.95	2.82	2.81	3.05

(a) Use a normal probability plot to check the normality assumption.
(b) Find a 99% two-sided confidence interval on the mean percentage of enrichment. Are you comfortable with the statement that the mean percentage of enrichment is 2.95 percent? Why?

8-31. A postmix beverage machine is adjusted to release a certain amount of syrup into a chamber where it is mixed with carbonated water. A random sample of 25 beverages was found to have a mean syrup content of $\bar{x} = 1.10$ fluid ounces and a standard deviation of $s = 0.015$ fluid ounces. Find a 95% CI on the mean volume of syrup dispensed.

8-32. An article in the *Journal of Composite Materials* (December 1989, Vol 23, p. 1200) describes the effect of delamination on the natural frequency of beams made from composite laminates. Five such delaminated beams were subjected to loads, and the resulting frequencies were as follows (in hertz):

230.66, 233.05, 232.58, 229.48, 232.58

Find a 90% two-sided confidence interval on mean natural frequency. Is there evidence to support the assumption of normality in the population?

8-4 CONFIDENCE INTERVAL ON THE VARIANCE AND STANDARD DEVIATION OF A NORMAL POPULATION

Sometimes confidence intervals on the population variance or standard deviation are needed. When the population is modeled by a normal distribution, the tests and intervals described in this section are applicable. The following result provides the basis of constructing these confidence intervals.

Definition

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2, and let S^2 be the sample variance. Then the random variable

$$X^2 = \frac{(n-1)S^2}{\sigma^2} \tag{8-19}$$

has a chi-square (χ^2) distribution with $n-1$ degrees of freedom.

The probability density function of a χ^2 random variable is

$$f(x) = \frac{1}{2^{k/2}\Gamma(k/2)} x^{(k/2)-1} e^{-x/2} \qquad x > 0 \tag{8-20}$$

where k is the number of degrees of freedom. The mean and variance of the χ^2 distribution are k and $2k$, respectively. Several chi-square distributions are shown in Fig. 8-8. Note that the chi-square random variable is nonnegative and that the probability distribution is skewed to the right. However, as k increases, the distribution becomes more symmetric. As $k \to \infty$, the limiting form of the chi-square distribution is the normal distribution.

The **percentage points** of the χ^2 distribution are given in Table III of the Appendix. Define $\chi^2_{\alpha,k}$ as the percentage point or value of the chi-square random variable with k degrees of freedom such that the probability that X^2 exceeds this value is α. That is,

$$P(X^2 > \chi^2_{\alpha,k}) = \int_{\chi^2_{\alpha,k}}^{\infty} f(u)\, du = \alpha$$

This probability is shown as the shaded area in Fig. 8-9(a). To illustrate the use of Table III, note that the areas α are the column headings and the degrees of freedom k are given in the left column. Therefore, the value with 10 degrees of freedom having an area (probability) of 0.05 to the right is $\chi^2_{0.05,10} = 18.31$. This value is often called an **upper** 5% point of chi-square with

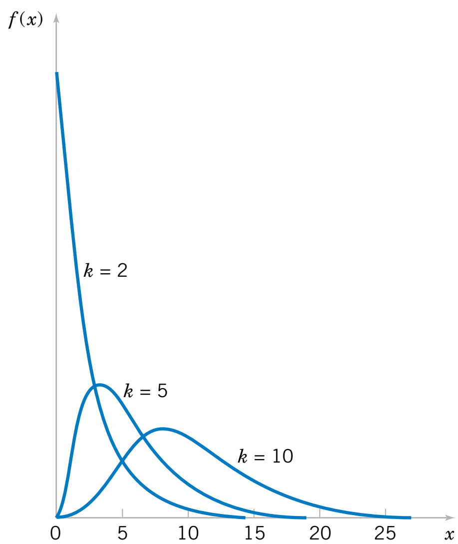


Figure 8-8 Probability density functions of several χ^2 distributions.

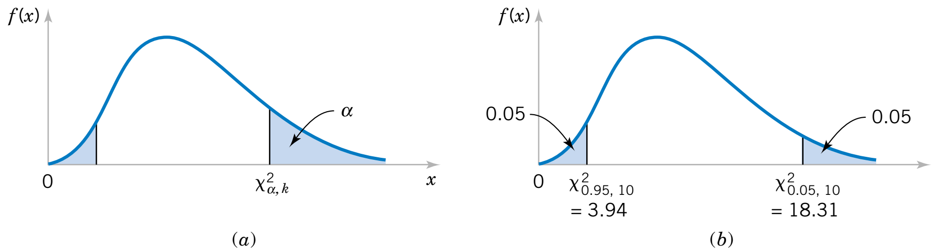


Figure 8-9 Percentage point of the χ^2 distribution. (a) The percentage point $\chi^2_{\alpha,k}$. (b) The upper percentage point $\chi^2_{0.05,10} = 18.31$ and the lower percentage point $\chi^2_{0.95,10} = 3.94$.

10 degrees of freedom. We may write this as a probability statement as follows:

$$P(X^2 > \chi^2_{0.05,10}) = P(X^2 > 18.31) = 0.05$$

Conversely, a **lower** 5% point of chi-square with 10 degrees of freedom would be $\chi^2_{0.95,10} = 3.94$ (from Appendix Table III). Both of these percentage points are shown in Figure 8-9(b).

The construction of the $100(1 - \alpha)\%$ CI for σ^2 is straightforward. Because

$$X^2 = \frac{(n-1)S^2}{\sigma^2}$$

is chi-square with $n - 1$ degrees of freedom, we may write

$$P(\chi^2_{1-\alpha/2,n-1} \le X^2 \le \chi^2_{\alpha/2,n-1}) = 1 - \alpha$$

so that

$$P\left(\chi^2_{1-\alpha/2,n-1} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{\alpha/2,n-1}\right) = 1 - \alpha$$

This last equation can be rearranged as

$$P\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}\right) = 1 - \alpha$$

This leads to the following definition of the confidence interval for σ^2.

Definition

If s^2 is the sample variance from a random sample of n observations from a normal distribution with unknown variance σ^2, then **a $100(1 - \alpha)\%$ confidence interval on σ^2 is**

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} \qquad (8\text{-}21)$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower $100\alpha/2$ percentage points of the chi-square distribution with $n - 1$ degrees of freedom, respectively. A **confidence interval for σ** has lower and upper limits that are the square roots of the corresponding limits in Equation 8-21.

It is also possible to find a $100(1-\alpha)\%$ lower confidence bound or upper confidence bound on σ^2.

The $100(1-\alpha)\%$ lower and upper confidence bounds on σ^2 are

$$\frac{(n-1)s^2}{\chi^2_{\alpha,n-1}} \leq \sigma^2 \quad \text{and} \quad \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}} \tag{8-22}$$

respectively.

EXAMPLE 8-5

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounces)^2. If the variance of fill volume is too large, an unacceptable proportion of bottles will be under- or overfilled. We will assume that the fill volume is approximately normally distributed. A 95% upper-confidence interval is found from Equation 8-22 as follows:

$$\sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{0.95,19}}$$

or

$$\sigma^2 \leq \frac{(19)0.0153}{10.117} = 0.0287 \text{ (fluid ounce)}^2$$

This last expression may be converted into a confidence interval on the standard deviation σ by taking the square root of both sides, resulting in

$$\sigma \leq 0.17$$

Therefore, at the 95% level of confidence, the data indicate that the process standard deviation could be as large as 0.17 fluid ounce.

EXERCISES FOR SECTION 8-4

8-33. Determine the values of the following percentiles: $\chi^2_{0.05,10}$, $\chi^2_{0.025,15}$, $\chi^2_{0.01,12}$, $\chi^2_{0.95,20}$, $\chi^2_{0.99,18}$, $\chi^2_{0.995,16}$, and $\chi^2_{0.005,25}$.

8-34. Determine the χ^2 percentile that is required to construct each of the following CIs:

(a) Confidence level = 95%, degrees of freedom = 24, one-sided (upper)

(b) Confidence level = 99%, degrees of freedom = 9, one-sided (lower)

(c) Confidence level = 90%, degrees of freedom = 19, two-sided.

8-35. A rivet is to be inserted into a hole. A random sample of $n = 15$ parts is selected, and the hole diameter is measured. The sample standard deviation of the hole diameter measurements is $s = 0.008$ millimeters. Construct a 99% lower confidence bound for σ^2.

8-36. The sugar content of the syrup in canned peaches is normally distributed. A random sample of $n = 10$ cans yields a sample standard deviation of $s = 4.8$ milligrams. Find a 95% two-sided confidence interval for σ.

8-37. Consider the tire life data in Exercise 8-22. Find a 95% lower confidence bound for σ^2.

8-38. Consider the Izod impact test data in Exercise 8-23. Find a 99% two-sided confidence interval for σ^2.

8-39. The percentage of titanium in an alloy used in aerospace castings is measured in 51 randomly selected parts. The sample standard deviation is $s = 0.37$. Construct a 95% two-sided confidence interval for σ.

8-40. Consider the hole diameter data in Exercise 8-35. Construct a 99% two-sided confidence interval for σ.

8-41. Consider the sugar content data in Exercise 8-37. Find a 90% lower confidence bound for σ.

8-5 A LARGE-SAMPLE CONFIDENCE INTERVAL FOR A POPULATION PROPORTION

It is often necessary to construct confidence intervals on a population proportion. For example, suppose that a random sample of size n has been taken from a large (possibly infinite) population and that $X(\leq n)$ observations in this sample belong to a class of interest. Then $\hat{P} = X/n$ is a point estimator of the proportion of the population p that belongs to this class. Note that n and p are the parameters of a binomial distribution. Furthermore, from Chapter 4 we know that the sampling distribution of $\hat{P}$ is approximately normal with mean p and variance $p(1-p)/n$, if p is not too close to either 0 or 1 and if n is relatively large. Typically, to apply this approximation we require that np and $n(1-p)$ be greater than or equal to 5. We will make use of the normal approximation in this section.

Definition

If n is large, the distribution of

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\hat{P} - p}{\sqrt{\dfrac{p(1-p)}{n}}}$$

is approximately standard normal.

To construct the confidence interval on p, note that

$$P(-z_{\alpha/2} \leq Z \leq z_{\alpha/2}) \simeq 1 - \alpha$$

so

$$P\left(-z_{\alpha/2} \leq \frac{\hat{P} - p}{\sqrt{\dfrac{p(1-p)}{n}}} \leq z_{\alpha/2}\right) \simeq 1 - \alpha$$

This may be rearranged as

$$P\left(\hat{P} - z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}} \leq p \leq \hat{P} + z_{\alpha/2}\sqrt{\frac{p(1-p)}{n}}\right) \simeq 1 - \alpha \tag{8-23}$$

The quantity $\sqrt{p(1-p)/n}$ in Equation 8-23 is called the **standard error of the point estimator** $\hat{P}$. Unfortunately, the upper and lower limits of the confidence interval obtained from

Equation 8-23 contain the unknown parameter p. However, as suggested at the end of Section 8-2.5, a satisfactory solution is to replace p by $\hat{P}$ in the standard error, which results in

$$P\left(\hat{P} - z_{\alpha/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} \le p \le \hat{P} + z_{\alpha/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}\right) \simeq 1-\alpha \qquad (8\text{-}24)$$

This leads to the approximate $100(1-\alpha)\%$ confidence interval on p.

Definition

If $\hat{p}$ is the proportion of observations in a random sample of size n that belongs to a class of interest, an approximate $100(1-\alpha)\%$ confidence interval on the proportion p of the population that belongs to this class is

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad (8\text{-}25)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

This procedure depends on the adequacy of the normal approximation to the binomial. To be reasonably conservative, this requires that np and $n(1-p)$ be greater than or equal to 5. In situations where this approximation is inappropriate, particularly in cases where n is small, other methods must be used. Tables of the binomial distribution could be used to obtain a confidence interval for p. However, we could also use numerical methods based on the binomial probability mass function that are implemented in computer programs.

EXAMPLE 8-6

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow. Therefore, a point estimate of the proportion of bearings in the population that exceeds the roughness specification is $\hat{p} = x/n = 10/85 = 0.12$. A 95% two-sided confidence interval for p is computed from Equation 8-25 as

$$\hat{p} - z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

or

$$0.12 - 1.96\sqrt{\frac{0.12(0.88)}{85}} \le p \le 0.12 + 1.96\sqrt{\frac{0.12(0.88)}{85}}$$

which simplifies to

$$0.05 \le p \le 0.19$$

Choice of Sample Size

Since $\hat{P}$ is the point estimator of p, we can define the error in estimating p by $\hat{P}$ as $E = |p - \hat{P}|$. Note that we are approximately $100(1-\alpha)\%$ confident that this error is less than $z_{\alpha/2}\sqrt{p(1-p)/n}$. For instance, in Example 8-6, we are 95% confident that the sample proportion $\hat{p} = 0.12$ differs from the true proportion p by an amount not exceeding 0.07.

In situations where the sample size can be selected, we may choose n to be $100(1-\alpha)\%$ confident that the error is less than some specified value E. If we set $E = z_{\alpha/2}\sqrt{p(1-p)/n}$ and solve for n, the appropriate sample size is

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1 - p) \tag{8-26}$$

An estimate of p is required to use Equation 8-26. If an estimate $\hat{p}$ from a previous sample is available, it can be substituted for p in Equation 8-26, or perhaps a subjective estimate can be made. If these alternatives are unsatisfactory, a preliminary sample can be taken, $\hat{p}$ computed, and then Equation 8-26 used to determine how many additional observations are required to estimate p with the desired accuracy. Another approach to choosing n uses the fact that the sample size from Equation 8-26 will always be a maximum for $p = 0.5$ [that is, $p(1 - p) \le 0.25$ with equality for $p = 0.5$], and this can be used to obtain an upper bound on n. In other words, we are at least $100(1 - \alpha)\%$ confident that the error in estimating p by $\hat{p}$ is less than E if the sample size is

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25) \tag{8-27}$$

EXAMPLE 8-7

Consider the situation in Example 8-6. How large a sample is required if we want to be 95% confident that the error in using $\hat{p}$ to estimate p is less than 0.05? Using $\hat{p} = 0.12$ as an initial estimate of p, we find from Equation 8-26 that the required sample size is

$$n = \left(\frac{z_{0.025}}{E}\right)^2 \hat{p}(1 - \hat{p}) = \left(\frac{1.96}{0.05}\right)^2 0.12(0.88) \cong 163$$

If we wanted to be *at least* 95% confident that our estimate $\hat{p}$ of the true proportion p was within 0.05 regardless of the value of p, we would use Equation 8-27 to find the sample size

$$n = \left(\frac{z_{0.025}}{E}\right)^2 (0.25) = \left(\frac{1.96}{0.05}\right)^2 (0.25) \cong 385$$

Notice that if we have information concerning the value of p, either from a preliminary sample or from past experience, we could use a smaller sample while maintaining both the desired precision of estimation and the level of confidence.

One-Sided Confidence Bounds

We may find approximate one-sided confidence bounds on p by a simple modification of Equation 8-25.

The approximate $100(1 - \alpha)\%$ lower and upper confidence bounds are

$$\hat{p} - z_{\alpha}\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \le p \qquad \text{and} \qquad p \le \hat{p} + z_{\alpha}\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}} \tag{8-28}$$

respectively.

EXERCISES FOR SECTION 8-5

8-42. Of 1000 randomly selected cases of lung cancer, 823 resulted in death within 10 years. Construct a 95% two-sided confidence interval on the death rate from lung cancer.

8-43. How large a sample would be required in Exercise 8-42 to be at least 95% confident that the error in estimating the 10-year death rate from lung cancer is less than 0.03?

8-44. A random sample of 50 suspension helmets used by motorcycle riders and automobile race-car drivers was subjected to an impact test, and on 18 of these helmets some damage was observed.

(a) Find a 95% two-sided confidence interval on the true proportion of helmets of this type that would show damage from this test.

(b) Using the point estimate of p obtained from the preliminary sample of 50 helmets, how many helmets must be tested to be 95% confident that the error in estimating the true value of p is less than 0.02?

(c) How large must the sample be if we wish to be at least 95% confident that the error in estimating p is less than 0.02, regardless of the true value of p?

8-45. The Arizona Department of Transportation wishes to survey state residents to determine what proportion of the population would like to increase statewide highway speed limits to 75 mph from 65 mph. How many residents do they need to survey if they want to be at least 99% confident that the sample proportion is within 0.05 of the true proportion?

8-46. A manufacturer of electronic calculators is interested in estimating the fraction of defective units produced. A random sample of 800 calculators contains 10 defectives. Compute a 99% upper-confidence bound on the fraction defective.

8-47. A study is to be conducted of the percentage of homeowners who own at least two television sets. How large a sample is required if we wish to be 99% confident that the error in estimating this quantity is less than 0.017?

8-48. The fraction of defective integrated circuits produced in a photolithography process is being studied. A random sample of 300 circuits is tested, revealing 13 defectives. Find a 95% two-sided CI on the fraction of defective circuits produced by this particular tool.

8-6 A PREDICTION INTERVAL FOR A FUTURE OBSERVATION

In some problem situations, we may be interested in **predicting** a future observation of a variable. This is a different problem than estimating the mean of that variable, so a confidence interval is not appropriate. In this section we show how to obtain a $100(1-\alpha)\%$ **prediction interval** on a future value of a normal random variable.

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal population. We wish to predict the value X_{n+1}, a single **future** observation. A point prediction of X_{n+1} is $\overline{X}$, the sample mean. The prediction error is $X_{n+1} - \overline{X}$. The expected value of the prediction error is

$$E(X_{n+1} - \overline{X}) = \mu - \mu = 0$$

and the variance of the prediction error is

$$V(X_{n+1} - \overline{X}) = \sigma^2 + \frac{\sigma^2}{n} = \sigma^2\left(1 + \frac{1}{n}\right)$$

because the future observation, X_{n+1} is independent of the mean of the current sample $\overline{X}$. The prediction error $X_{n+1} - \overline{X}$ is normally distributed. Therefore

$$Z = \frac{X_{n+1} - \overline{X}}{\sigma\sqrt{1 + \dfrac{1}{n}}}$$

has a standard normal distribution. Replacing σ with S results in

$$T = \frac{X_{n+1} - \overline{X}}{S\sqrt{1 + \frac{1}{n}}}$$

which has a t distribution with $n - 1$ degrees of freedom. Manipulating T as we have done previously in the development of a CI leads to a prediction interval on the future observation X_{n+1}.

Definition

A 100(1 − α)% prediction interval on a single future observation from a normal distribution is given by

$$\bar{x} - t_{\alpha/2,n-1}\, s\sqrt{1 + \frac{1}{n}} \le X_{n+1} \le \bar{x} + t_{\alpha/2,n-1}\, s\sqrt{1 + \frac{1}{n}} \qquad (8\text{-}29)$$

The prediction interval for X_{n+1} will always be longer than the confidence interval for μ because there is more variability associated with the prediction error than with the error of estimation. This is easy to see because the prediction error is the difference between two random variables $(X_{n+1} - \overline{X})$, and the estimation error in the CI is the difference between one random variable and a constant $(\overline{X} - \mu)$. As n gets larger $(n \to \infty)$, the length of the CI decreases to zero, essentially becoming the single value μ, but the length of the prediction interval approaches $2z_{\alpha/2}\sigma$. So as n increases, the uncertainty in estimating μ goes to zero, although there will always be uncertainty about the future value X_{n+1} even when there is no need to estimate any of the distribution parameters.

EXAMPLE 8-8

Reconsider the tensile adhesion tests on specimens of U-700 alloy described in Example 8-4. The load at failure for $n = 22$ specimens was observed, and we found that $\bar{x} = 13.71$ and $s = 3.55$. The 95% confidence interval on μ was $12.14 \le \mu \le 15.28$. We plan to test a twenty-third specimen. A 95% prediction interval on the load at failure for this specimen is

$$\bar{x} - t_{\alpha/2,n-1}\, s\sqrt{1 + \frac{1}{n}} \le X_{n+1} \le \bar{x} + t_{\alpha/2,n-1}\, s\sqrt{1 + \frac{1}{n}}$$

$$13.71 - (2.080)3.55\sqrt{1 + \frac{1}{22}} \le X_{23} \le 13.71 + (2.080)3.55\sqrt{1 + \frac{1}{22}}$$

$$6.16 \le X_{23} \le 21.26$$

Notice that the prediction interval is considerably longer than the CI.

EXERCISES FOR SECTION 8-6

8-49. Consider the tire-testing data described in Exercise 8-22. Compute a 95% prediction interval on the life of the next tire of this type tested under conditions that are similar to those employed in the original test. Compare the length of the prediction interval with the length of the 95% CI on the population mean.

8-50. Consider the Izod impact test described in Exercise 8-23. Compute a 99% prediction interval on the impact strength of the next specimen of PVC pipe tested. Compare the length of the prediction interval with the length of the 99% CI on the population mean.

8-51. Consider the television tube brightness test described in Exercise 8-24. Compute a 99% prediction interval on the brightness of the next tube tested. Compare the length of the prediction interval with the length of the 99% CI on the population mean.

8-52. Consider the margarine test described in Exercise 8-25. Compute a 99% prediction interval on the polyunsaturated fatty acid in the next package of margarine that is tested. Compare the length of the prediction interval with the length of the 99% CI on the population mean.

8-53. Consider the test on the compressive strength of concrete described in Exercise 8-26. Compute a 90% prediction interval on the next specimen of concrete tested.

8-54. Consider the suspension rod diameter measurements described in Exercise 8-27. Compute a 95% prediction interval on the diameter of the next rod tested. Compare the length of the prediction interval with the length of the 95% CI on the population mean.

8-55. Consider the bottle wall thickness measurements described in Exercise 8-29. Compute a 90% prediction interval on the wall thickness of the next bottle tested.

8-56. How would you obtain a one-sided prediction bound on a future observation? Apply this procedure to obtain a 95% one-sided prediction bound on the wall thickness of the next bottle for the situation described in Exercise 8-29.

8-57. Consider the fuel rod enrichment data described in Exercise 8-30. Compute a 99% prediction interval on the enrichment of the next rod tested. Compare the length of the prediction interval with the length of the 95% CI on the population mean.

8-58. Consider the syrup dispensing measurements described in Exercise 8-31. Compute a 95% prediction interval on the syrup volume in the next beverage dispensed. Compare the length of the prediction interval with the length of the 95% CI on the population mean.

8-59. Consider the natural frequency of beams described in Exercise 8-32. Compute a 90% prediction interval on the diameter of the natural frequency of the next beam of this type that will be tested. Compare the length of the prediction interval with the length of the 95% CI on the population mean.

8-7 TOLERANCE INTERVALS FOR A NORMAL DISTRIBUTION

Consider a population of semiconductor processors. Suppose that the speed of these processors has a normal distribution with mean $\mu = 600$ megahertz and standard deviation $\sigma = 30$ megahertz. Then the interval from $600 - 1.96(30) = 541.2$ to $600 + 1.96(30) = 658.8$ megahertz captures the speed of 95% of the processors in this population because the interval from -1.96 to 1.96 captures 95% of the area under the standard normal curve. The interval from $\mu - z_{\alpha/2}$ to $\mu + z_{\alpha/2}\sigma$ is called a **tolerance interval.**

If μ and σ are unknown, we can use the data from a random sample of size n to compute $\bar{x}$ and s, and then form the interval $(\bar{x} - 1.96s, \bar{x} + 1.96s)$. However, because of sampling variability in $\bar{x}$ and s, it is likely that this interval will contain less than 95% of the values in the population. The solution to this problem is to replace 1.96 by some value that will make the proportion of the distribution contained in the interval 95% with some level of confidence. Fortunately, it is easy to do this.

Definition

A **tolerance interval** for capturing at least γ% of the values in a normal distribution with confidence level $100(1 - \alpha)$% is

$$\bar{x} - ks, \qquad \bar{x} + ks$$

where k is a tolerance interval factor found in Appendix Table XI. Values are given for $\gamma = 90\%$, 95%, and 95% and for 95% and 99% confidence.

One-sided tolerance bounds can also be computed. The tolerance factors for these bounds are also given in Appendix Table XI.

EXAMPLE 8-9

Let's reconsider the tensile adhesion tests originally described in Example 8-4. The load at failure for $n = 22$ specimens was observed, and we found that $\bar{x} = 31.71$ and $s = 3.55$. We want to find a tolerance interval for the load at failure that includes 90% of the values in the population with 95% confidence. From Appendix Table XI the tolerance factor k for $n = 22$, $\gamma = 0.90$, and 95% confidence is $k = 2.264$. The desired tolerance interval is

$$(\bar{x} - ks, \bar{x} + ks) \quad \text{or} \quad [31.71 - (2.264)3.55, 31.71 + (2.264)3.55]$$

which reduces to (23.67, 39.75). We can be 95% confident that at least 90% of the values of load at failure for this particular alloy lie between 23.67 and 39.75 megapascals.

From Appendix Table XI, we note that as $n \to \infty$, the value of k goes to the z-value associated with the desired level of containment for the normal distribution. For example, if we want 90% of the population to fall in the two-sided tolerance interval, k approaches $z_{0.05} = 1.645$ as $n \to \infty$. Note that as $n \to \infty$, a $100(1 - \alpha)\%$ prediction interval on a future value approaches a tolerance interval that contains $100(1 - \alpha)\%$ of the distribution.

EXERCISES FOR SECTION 8-7

8-60. Compute a 95% tolerance interval on the life of the tires described in Exercise 8-22, that has confidence level 95%. Compare the length of the tolerance interval with the length of the 95% CI on the population mean. Which interval is shorter? Discuss the difference in interpretation of these two intervals.

8-61. Consider the Izod impact test described in Exercise 8-23. Compute a 99% tolerance interval on the impact strength of PVC pipe that has confidence level 90%. Compare the length of the tolerance interval with the length of the 99% CI on the population mean. Which interval is shorter? Discuss the difference in interpretation of these two intervals.

8-62. Compute a 99% tolerance interval on the brightness of the television tubes in Exercise 8-24 that has confidence level 95%. Compare the length of the prediction interval with the length of the 99% CI on the population mean. Which interval is shorter? Discuss the difference in interpretation of these two intervals.

8-63. Consider the margarine test described in Exercise 8-25. Compute a 99% tolerance interval on the polyunsaturated fatty acid in this particular type of margarine that has confidence level 95%. Compare the length of the prediction interval with the length of the 99% CI on the population mean. Which interval is shorter? Discuss the difference in interpretation of these two intervals.

8-64. Compute a 90% tolerance interval on the compressive strength of the concrete described in Exercise 8-26 that has 90% confidence.

8-65. Compute a 95% tolerance interval on the diameter of the rods described in Exercise 8-27 that has 90% confidence. Compare the length of the prediction interval with the length of the 95% CI on the population mean. Which interval is shorter? Discuss the difference in interpretation of these two intervals.

8-66. Consider the bottle wall thickness measurements described in Exercise 8-29. Compute a 90% tolerance interval on bottle wall thickness that has confidence level 90%.

8-67. Consider the bottle wall thickness measurements described in Exercise 8-29. Compute a 90% lower tolerance bound on bottle wall thickness that has confidence level 90%. Why would a lower tolerance bound likely be of interest here?

8-68. Consider the fuel rod enrichment data described in Exercise 8-30. Compute a 99% tolerance interval on rod enrichment that has confidence level 95%. Compare the length of the prediction interval with the length of the 95% CI on the population mean.

8-69. Compute a 95% tolerance interval on the syrup volume described in Exercise 8-31 that has confidence level 90%. Compare the length of the prediction interval with the length of the 95% CI on the population mean.

Supplemental Exercises

8-70. Consider the confidence interval for μ with known standard deviation σ:

$$\bar{x} - z_{\alpha_1}\sigma/\sqrt{n} \leq \mu \leq \bar{x} + z_{\alpha_2}\sigma/\sqrt{n}$$

where $\alpha_1 + \alpha_2 = \alpha$. Let $\alpha = 0.05$ and find the interval for $\alpha_1 = \alpha_2 = \alpha/2 = 0.025$. Now find the interval for the case $\alpha_1 = 0.01$ and $\alpha_2 = 0.04$. Which interval is shorter? Is there any advantage to a "symmetric" confidence interval?

8-71. A normal population has a known mean 50 and unknown variance.

(a) A random sample of $n = 16$ is selected from this population, and the sample results are $\bar{x} = 52$ and $s = 8$. How unusual are these results? That is, what is the probability of observing a sample average as large as 52 (or larger) if the known, underlying mean is actually 50?

(b) A random sample of $n = 30$ is selected from this population, and the sample results are $\bar{x} = 52$ and $s = 8$. How unusual are these results?

(c) A random sample of $n = 100$ is selected from this population, and the sample results are $\bar{x} = 52$ and $s = 8$. How unusual are these results?

(d) Compare your answers to parts (a)–(c) and explain why they are the same or differ.

8-72. A normal population has known mean $\mu = 50$ and variance $\sigma^2 = 5$. What is the approximate probability that the sample variance is greater than or equal to 7.44? less than or equal to 2.56?

(a) For a random sample of $n = 16$.

(b) For a random sample of $n = 30$.

(c) For a random sample of $n = 71$.

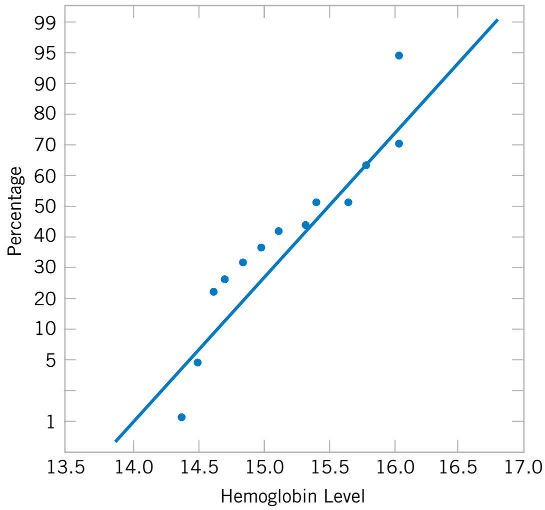
(d) Compare your answers to parts (a)–(c) for the approximate probability that the sample variance is greater than or equal to 7.44. Explain why this tail probability is increasing or decreasing with increased sample size.

(e) Compare your answers to parts (a)–(c) for the approximate probability that the sample variance is less than or equal to 2.56. Explain why this tail probability is increasing or decreasing with increased sample size.

8-73. An article in the *Journal of Sports Science* (1987, Vol. 5, pp. 261–271) presents the results of an investigation of the hemoglobin level of Canadian Olympic ice hockey players. The data reported are as follows (in g/dl):

15.3	16.0	14.4	16.2	16.2
14.9	15.7	15.3	14.6	15.7
16.0	15.0	15.7	16.2	14.7
14.8	14.6	15.6	14.5	15.2

(a) Given the following probability plot of the data, what is a logical assumption about the underlying distribution of the data?



(b) Explain why this check of the distribution underlying the sample data is important if we want to construct a confidence interval on the mean.

(c) Based on this sample data, a 95% confidence interval for the mean is (15.04, 15.62). Is it reasonable to infer that the true mean could be 14.5? Explain your answer.

(d) Explain why this check of the distribution underlying the sample data is important if we want to construct a confidence interval on the variance.

(e) Based on this sample data, a 95% confidence interval for the variance is (0.22, 0.82). Is it reasonable to infer that the true variance could be 0.35? Explain your answer.

(f) Is it reasonable to use these confidence intervals to draw an inference about the mean and variance of hemoglobin levels

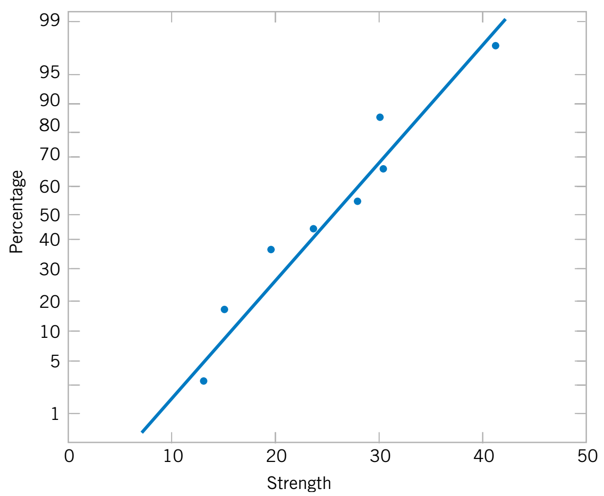
(i) of Canadian doctors? Explain your answer.

(ii) of Canadian children ages 6–12? Explain your answer.

8-74. The article "Mix Design for Optimal Strength Development of Fly Ash Concrete" (*Cement and Concrete Research*, 1989, Vol. 19, No. 4, pp. 634–640) investigates the compressive strength of concrete when mixed with fly ash (a mixture of silica, alumina, iron, magnesium oxide, and other ingredients). The compressive strength for nine samples in dry conditions on the twenty-eighth day are as follows (in megapascals):

40.2	30.4	28.9	30.5	22.4
25.8	18.4	14.2	15.3	

(a) Given the following probability plot of the data, what is a logical assumption about the underlying distribution of the data?



(b) Find a 99% lower one-sided confidence interval on mean compressive strength. Provide a practical interpretation of this interval.

(c) Find a 98% two-sided confidence interval on mean compressive strength. Provide a practical interpretation of this interval and explain why the lower end-point of the interval is or is not the same as in part (b).

(d) Find a 99% upper one-sided confidence interval on the variance of compressive strength. Provide a practical interpretation of this interval.

(e) Find a 98% two-sided confidence interval on the variance of compression strength. Provide a practical interpretation of this interval and explain why the upper end-point of the interval is or is not the same as in part (d).

(f) Suppose that it was discovered that the largest observation 40.2 was misrecorded and should actually be 20.4. Now the sample mean $\bar{x} = 23$ and the sample variance $s^2 = 36.9$. Use these new values and repeat parts (c) and (e). Compare the original computed intervals and the newly computed intervals with the corrected observation value. How does this mistake affect the values of the sample mean, sample variance, and the width of the two-sided confidence intervals?

(g) Suppose, instead, that it was discovered that the largest observation 40.2 is correct, but that the observation 25.8 is incorrect and should actually be 24.8. Now the sample mean $\bar{x} = 25$ and the sample variance $s^2 = 8.41$. Use these new values and repeat parts (c) and (e). Compare the original computed intervals and the newly computed intervals with the corrected observation value. How does this mistake affect the values of the sample mean, sample variance, and the width of the two-sided confidence intervals?

(h) Use the results from parts (f) and (g) to explain the effect of mistakenly recorded values on sample estimates. Comment on the effect when the mistaken values are near the sample mean and when they are not.

8-75. An operating system for a personal computer has been studied extensively, and it is known that the standard deviation of the response time following a particular command is $\sigma = 8$ milliseconds. A new version of the operating system is installed, and we wish to estimate the mean response time for the new system to ensure that a 95% confidence interval for μ has length at most 5 milliseconds.

(a) If we can assume that response time is normally distributed and that $\sigma = 8$ for the new system, what sample size would you recommend?

(b) Suppose that we are told by the vendor that the standard deviation of the response time of the new system is smaller, say $\sigma = 6$; give the sample size that you recommend and comment on the effect the smaller standard deviation has on this calculation.

8-76. Consider the hemoglobin data in Exercise 8-73. Find the following:

(a) An interval that contains 95% of the hemoglobin values with 90% confidence.

(b) An interval that contains 99% of the hemoglobin values with 90% confidence.

8-77. Consider the compressive strength of concrete data from Exercise 8-74. Find a 95% prediction interval on the next sample that will be tested.

8-78. The maker of a shampoo knows that customers like this product to have a lot of foam. Ten sample bottles of the product are selected at random and the foam heights observed are as follows (in millimeters): 210, 215, 194, 195, 211, 201, 198, 204, 208, and 196.

(a) Is there evidence to support the assumption that foam height is normally distributed?

(b) Find a 95% CI on the mean foam height.

(c) Find a 95% prediction interval on the next bottle of shampoo that will be tested.

(d) Find an interval that contains 95% of the shampoo foam heights with 99% confidence.

(e) Explain the difference in the intervals computed in parts (b), (c), and (d).

8-79. During the 1999 and 2000 baseball seasons, there was much speculation that the unusually large number of home runs that were hit was due at least in part to a livelier ball. One way to test the "liveliness" of a baseball is to launch the ball at a vertical surface with a known velocity V_L and measure the ratio of the outgoing velocity V_O of the ball to V_L. The ratio $R = V_O/V_L$ is called the coefficient of restitution. Following are measurements of the coefficient of restitution for 40 randomly selected baseballs. The balls were thrown from a pitching machine at an oak surface.

0.6248	0.6237	0.6118	0.6159	0.6298	0.6192
0.6520	0.6368	0.6220	0.6151	0.6121	0.6548
0.6226	0.6280	0.6096	0.6300	0.6107	0.6392
0.6230	0.6131	0.6223	0.6297	0.6435	0.5978

0.6351	0.6275	0.6261	0.6262	0.6262	0.6314
0.6128	0.6403	0.6521	0.6049	0.6170	
0.6134	0.6310	0.6065	0.6214	0.6141	

(a) Is there evidence to support the assumption that the coefficient of restitution is normally distributed?
(b) Find a 99% CI on the mean coefficient of restitution.
(c) Find a 99% prediction interval on the coefficient of restitution for the next baseball that will be tested.
(d) Find an interval that will contain 99% of the values of the coefficient of restitution with 95% confidence.
(e) Explain the difference in the three intervals computed in parts (b), (c), and (d).

8-80. Consider the baseball coefficient of restitution data in Exercise 8-79. Suppose that any baseball that has a coefficient of restitution that exceeds 0.635 is considered too lively. Based on the available data, what proportion of the baseballs in the sampled population are too lively? Find a 95% lower confidence bound on this proportion.

8-81. An article in the *ASCE Journal of Energy Engineering* ("Overview of Reservoir Release Improvements at 20 TVA Dams," Vol. 125, April 1999, pp. 1–17) presents data on dissolved oxygen concentrations in streams below 20 dams in the Tennessee Valley Authority system. The observations are (in milligrams per liter): 5.0, 3.4, 3.9, 1.3, 0.2, 0.9, 2.7, 3.7, 3.8, 4.1, 1.0, 1.0, 0.8, 0.4, 3.8, 4.5, 5.3, 6.1, 6.9, and 6.5.
(a) Is there evidence to support the assumption that the dissolved oxygen concentration is normally distributed?
(b) Find a 95% CI on the mean dissolved oxygen concentration.
(c) Find a 95% prediction interval on the dissolved oxygen concentration for the next stream in the system that will be tested.
(d) Find an interval that will contain 95% of the values of the dissolved oxygen concentration with 99% confidence.
(e) Explain the difference in the three intervals computed in parts (b), (c), and (d).

8-82. The tar content in 30 samples of cigar tobacco follows:

1.542	1.585	1.532	1.466	1.499	1.611
1.622	1.466	1.546	1.494	1.548	1.626
1.440	1.608	1.520	1.478	1.542	1.511
1.459	1.533	1.532	1.523	1.397	1.487
1.598	1.498	1.600	1.504	1.545	1.558

(a) Is there evidence to support the assumption that the tar content is normally distributed?
(b) Find a 99% CI on the mean tar content.
(c) Find a 99% prediction interval on the tar content for the next observation that will be taken on this particular type of tobacco.
(d) Find an interval that will contain 99% of the values of the tar content with 95% confidence.
(e) Explain the difference in the three intervals computed in parts (b), (c), and (d).

8-83. A manufacturer of electronic calculators takes a random sample of 1200 calculators and finds that there are eight defective units.
(a) Construct a 95% confidence interval on the population proportion.
(b) Is there evidence to support a claim that the fraction of defective units produced is 1% or less?

8-84. An article in *The Engineer* ("Redesign for Suspect Wiring," June 1990) reported the results of an investigation into wiring errors on commercial transport aircraft that may produce faulty information to the flight crew. Such a wiring error may have been responsible for the crash of a British Midland Airways aircraft in January 1989 by causing the pilot to shut down the wrong engine. Of 1600 randomly selected aircraft, eight were found to have wiring errors that could display incorrect information to the flight crew.
(a) Find a 99% confidence interval on the proportion of aircraft that have such wiring errors.
(b) Suppose we use the information in this example to provide a preliminary estimate of p. How large a sample would be required to produce an estimate of p that we are 99% confident differs from the true value by at most 0.008?
(c) Suppose we did not have a preliminary estimate of p. How large a sample would be required if we wanted to be at least 99% confident that the sample proportion differs from the true proportion by at most 0.008 regardless of the true value of p?
(d) Comment on the usefulness of preliminary information in computing the needed sample size.

8-85. An article in *Engineering Horizons* (Spring 1990, p. 26) reported that 117 of 484 new engineering graduates were planning to continue studying for an advanced degree. Consider this as a random sample of the 1990 graduating class.
(a) Find a 90% confidence interval on the proportion of such graduates planning to continue their education.
(b) Find a 95% confidence interval on the proportion of such graduates planning to continue their education.
(c) Compare your answers to parts (a) and (b) and explain why they are the same or different.
(d) Could you use either of these confidence intervals to determine whether the proportion is actually 0.25? Explain your answer. *Hint:* Use the normal approximation to the binomial.

MIND-EXPANDING EXERCISES

8-86. An electrical component has a time-to-failure (or lifetime) distribution that is exponential with parameter λ, so the mean lifetime is $\mu = 1/\lambda$. Suppose that a sample of n of these components is put on test, and let X_i be the observed lifetime of component i. The test continues only until the rth unit fails, where $r < n$. This results in a **censored** life test. Let X_1 denote the time at which the first failure occurred, X_2 denote the time at which the second failure occurred, and so on. Then the total lifetime that has been accumulated at test termination is

$$T_r = \sum_{i=1}^{r} X_i + (n - r)X_r$$

We have previously shown in Exercise 7-72 that T_r/r is an unbiased estimator for μ.

(a) It can be shown that $2\lambda T_r$ has a chi-square distribution with $2r$ degrees of freedom. Use this fact to develop a $100(1 - \alpha)\%$ confidence interval for mean lifetime $\mu = 1/\lambda$.

(b) Suppose 20 units were put on test, and the test terminated after 10 failures occurred. The failure times (in hours) are 15, 18, 19, 20, 21, 21, 22, 27, 28, 29. Find a 95% confidence interval on mean lifetime.

8-87. Consider a two-sided confidence interval for the mean μ when σ is known;

$$\bar{x} - z_{\alpha_1}\sigma/\sqrt{n} \le \mu \le \bar{x} + z_{\alpha_2}\sigma/\sqrt{n}$$

where $\alpha_1 + \alpha_2 = \alpha$. If $\alpha_1 = \alpha_2 = \alpha/2$, we have the usual $100(1 - \alpha)\%$ confidence interval for μ. In the above, when $\alpha_1 \ne \alpha_2$, the interval is not symmetric about μ. The length of the interval is $L = \sigma(z_{\alpha_1} + z_{\alpha_2})/\sqrt{n}$. Prove that the length of the interval L is minimized when $\alpha_1 = \alpha_2 = \alpha/2$. *Hint:* Remember that $\Phi(z_a) = 1 - \alpha$, so $\Phi^{-1}(1 - \alpha) = z_\alpha$, and the relationship between the derivative of a function $y = f(x)$ and the inverse $x = f^{-1}(y)$ is $(d/dy)f^{-1}(y) = 1/[(d/dx)f(x)]$.

8-88. It is possible to construct a **nonparametric tolerance interval** that is based on the extreme values in a random sample of size n from any continuous population. If p is the minimum proportion of the population contained between the smallest and largest sample observations with confidence $1 - \alpha$, it can be shown that

$$np^{n-1} - (n - 1)p^n = \alpha$$

and n is approximately

$$n = \frac{1}{2} + \left(\frac{1 + p}{1 - p}\right)\left(\frac{\chi^2_{\alpha,4}}{4}\right)$$

(a) In order to be 95% confident that at least 90% of the population will be included between the extreme values of the sample, what sample size will be required?

(b) A random sample of 10 transistors gave the following measurements on saturation current (in milliamps): 10.25, 10.41, 10.30, 10.26, 10.19, 10.37, 10.29, 10.34, 10.23, 10.38. Find the limits that contain a proportion p of the saturation current measurements at 95% confidence. What is the proportion p contained by these limits?

8-89. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a continuous probability distribution with median $\tilde{\mu}$.

(a) Show that

$$P\{\min(X_i) < \tilde{\mu} < \max(X_i)\} = 1 - \left(\frac{1}{2}\right)^{n-1}$$

Hint: The complement of the event $[\min(X_i) < \tilde{\mu} < \max(X_i)]$ is $[\max(X_i) \le \tilde{\mu}] \cup [\min(X_i) \le \tilde{\mu}]$, but $\max(X_i) \le \tilde{\mu}$ if and only if $X_i \le \tilde{\mu}$ for all i.]

(b) Write down a $100(1 - \alpha)\%$ confidence interval for the median $\tilde{\mu}$, where

$$\alpha = \left(\frac{1}{2}\right)^{n-1}.$$

8-90. Students in the industrial statistics lab at ASU calculate a lot of confidence intervals on μ. Suppose all these CIs are independent of each other. Consider the next one thousand 95% confidence intervals that will be calculated. How many of these CIs do you expect to capture the true value of μ? What is the probability that between 930 and 970 of these intervals contain the true value of μ?

IMPORTANT TERMS AND CONCEPTS

In the E-book, click on any term or concept below to go to that subject.

Confidence coefficient
Confidence interval
Confidence interval for a population proportion
Chi-squared distribution
Confidence intervals for the mean of a normal distribution
Confidence interval for the variance of a normal distribution
Confidence level
Error in estimation
Large sample confidence intervals
One-sided confidence bounds
Precision of parameter estimation
Prediction interval
Tolerance interval
Two-sided confidence interval
***t* distribution**

CD MATERIAL

Boostrap samples
Percentile method for boostrap confidence intervals

8-2.6 Bootstrap Confidence Intervals (CD Only)

In Section 7-2.5 we showed how a technique called the **bootstrap** could be used to estimate the standard error $\sigma_{\hat{\theta}}$, where $\hat{\theta}$ is an estimate of a parameter θ. We can also use the bootstrap to find a confidence interval on the parameter θ. To illustrate, consider the case where θ is the mean μ of a normal distribution with σ known. Now the estimator of θ is $\overline{X}$. Also notice that $z_{\alpha/2}\sigma/\sqrt{n}$ is the $100(1-\alpha/2)$ percentile of the distribution of $\overline{X}-\mu$, and $-z_{\alpha/2}\sigma/\sqrt{n}$ is the $100(\alpha/2)$ percentile of this distribution. Therefore, we can write the probability statement associated with the $100(1-\alpha)$% confidence interval as

$$P(100(\alpha/2) \text{ percentile} \leq \overline{X}-\mu \leq 100(1-\alpha/2) \text{ percentile}) = 1-\alpha$$

or

$$P(\overline{X} - 100(1-\alpha/2) \text{ percentile} \leq \mu \leq \overline{X} - 100(\alpha/2) \text{ percentile}) = 1-\alpha$$

This last probability statement implies that the lower and upper $100(1-\alpha)$% confidence limits for μ are

$$L = \overline{X} - 100(1-\alpha/2) \text{ percentile of } \overline{X}-\mu = \overline{X} - z_{\alpha/2}\sigma/\sqrt{n}$$
$$U = \overline{X} - 100(\alpha/2) \text{ percentile of } \overline{X}-\mu = \overline{X} + z_{\alpha/2}\sigma/\sqrt{n}$$

We may generalize this to an arbitrary parameter θ. The $100(1-\alpha)$% confidence limits for θ are

$$L = \hat{\theta} - 100(1-\alpha/2) \text{ percentile of } \hat{\theta}-\theta$$
$$U = \hat{\theta} - 100(\alpha/2) \text{ percentile of } \hat{\theta}-\theta$$

Unfortunately, the percentiles of $\hat{\theta}-\theta$ may not be as easy to find as in the case of the normal distribution mean. However, they could be estimated from **bootstrap samples.** Suppose we find B bootstrap samples and calculate $\hat{\theta}_1^*, \hat{\theta}_2^*, \ldots, \hat{\theta}_B^*$ and $\overline{\theta}^*$ and then calculate $\hat{\theta}_1^* - \overline{\theta}^*$, $\hat{\theta}_2^* - \overline{\theta}^*, \ldots, \hat{\theta}_B^* - \overline{\theta}^*$. The required percentiles can be obtained directly from the differences. For example, if $B = 200$ and a 95% confidence interval on θ is desired, the fifth smallest and fifth largest of the differences $\hat{\theta}_i^* - \overline{\theta}^*$ are the estimates of the necessary percentiles.

We will illustrate this procedure using the situation first described in Example 7-3, involving the parameter λ of an exponential distribution. Following that example, a random sample of $n = 8$ engine controller modules were tested to failure, and the estimate of λ obtained was $\hat{\lambda} = 0.0462$, where $\hat{\lambda} = 1/\overline{X}$ is a maximum likelihood estimator. We used 200 bootstrap samples to obtain an estimate of the standard error for $\hat{\lambda}$.

Figure S8-1(a) is a histogram of the 200 bootstrap estimates $\hat{\lambda}_i^*$, $i = 1, 2, \ldots, 200$. Notice that the histogram is not symmetrical and is skewed to the right, indicating that the sampling distribution of $\hat{\lambda}$ also has this same shape. We subtracted the sample average of these bootstrap estimates $\overline{\lambda}^* = 0.5013$ from each $\hat{\lambda}_i^*$. The histogram of the differences $\hat{\lambda}_i^* - \overline{\lambda}^*$, $i = 1, 2, \ldots, 200$, is shown in Figure S8-1(b). Suppose we wish to find a 90% confidence interval for λ. Now the fifth percentile of the bootstrap samples $\hat{\lambda}_i^* - \overline{\lambda}^*$ is -0.0228 and the ninety-fifth percentile is 0.03135. Therefore the lower and upper 90% bootstrap confidence limits are

$$L = \hat{\lambda} - 95 \text{ percentile of } \hat{\lambda}_i^* - \overline{\lambda}^* = 0.0462 - 0.03135 = 0.0149$$
$$U = \hat{\lambda} - 5 \text{ percentile of } \hat{\lambda}_i^* - \overline{\lambda}^* = 0.0462 - (-0.0228) = 0.0690$$

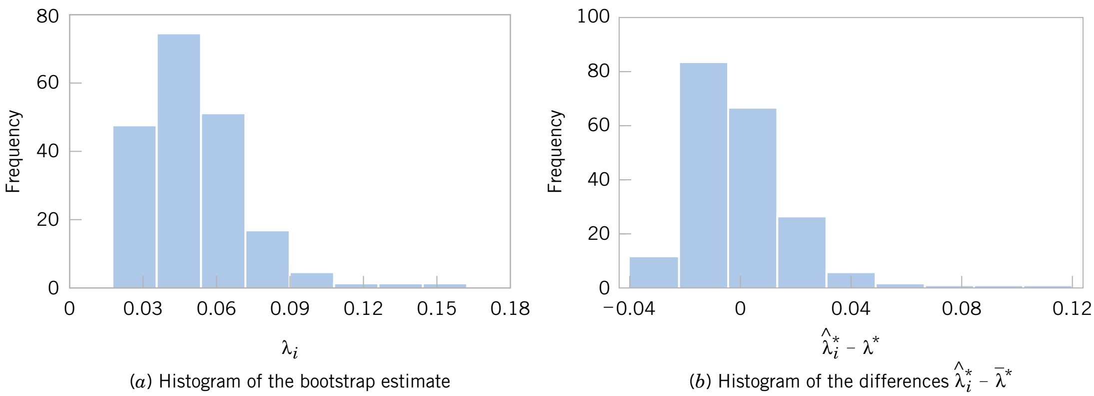


Figure S8-1 Histograms of the bootstrap estimates of λ and the differences $\hat{\lambda}_i^* - \hat{\lambda}^*$ used in finding the bootstrap confidence interval.

Therefore, our 90% bootstrap confidence interval for λ is $0.0149 \le \lambda \le 0.0690$. There is an exact confidence interval for the parameter λ in an exponential distribution. For the engine controller failure data following Example 7-3, the exact 90% confidence interval* for λ is $0.0230 \le \lambda \le 0.0759$. Notice that the two confidence intervals are very similar. The length of the exact confidence interval is $0.0759 - 0.0230 = 0.0529$, while the length of the bootstrap confidence interval is $0.0690 - 0.0149 = 0.0541$, which is only slightly longer. The percentile method for bootstrap confidence intervals works well when the estimator is unbiased and the standard error of $\hat{\theta}$ is approximately constant (as a function of θ). An improvement, known as the *bias-corrected and accelerated* method, adjusts the percentiles in more general cases. It could be applied in this example (because $\hat{\lambda}$ is a biased estimator), but at the cost of additional complexity.

8-3.2 Development of the *t*-Distribution (CD Only)

We will give a formal development of the *t*-distribution using the techniques presented in Section 5-8. It will be helpful to review that material before reading this section.

First consider the random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$$

This quantity can be written as

$$T = \frac{\dfrac{\overline{X} - \mu}{\sigma\sqrt{n}}}{\sqrt{S^2/\sigma^2}} \qquad \text{(S8-1)}$$

*The confidence interval is $\chi^2_{\alpha/2,2n}/(2\sum x_i) \le \lambda \le \chi^2_{1-\alpha/2,2n}/(2\sum x_i)$ where $\chi^2_{\alpha/2,2n}$ and $\chi^2_{1-\alpha/2,2n}$ are the lower and upper $\alpha/2$ percentage points of the chi-square distribution (which was introduced briefly in Chapter 4 and discussed further in Section 8-4), and the x_i are the n sample observations.

Now the numerator of Equation S8-1 is a standard normal random variable. The ratio S^2/σ^2 in the denominator is a chi-square random variable with $n - 1$ degrees of freedom, divided by the number of degrees of freedom*. Furthermore, the two random variables in Equation S8-1, $\overline{X}$ and S, are independent. We are now ready to state and prove the main result.

Theorem S8-1: The *t*-Distribution

Let Z be a standard normal random variable and V be a chi-square random variable with k degrees of freedom. If Z and V are independent, the distribution of the random variable

$$T = \frac{Z}{\sqrt{V/k}}$$

is the t-distribution with k degrees of freedom. The probability density function is

$$f(t) = \frac{\Gamma[(k+1)/2]}{\sqrt{\pi k}\,\Gamma(k/2)} \frac{1}{[(t^2/k)+1]^{(k+1)/2}}, \quad -\infty < t < \infty$$

Proof Since Z and V are independent, their probability distribution is

$$f_{ZV}(z, v) = \frac{v^{(k/2)-1}}{\sqrt{2\pi}2^{k/2}\Gamma\left(\frac{k}{2}\right)} e^{-(z^2+v)/2}, \qquad -\infty < z < \infty, 0 < v < \infty$$

Define a new random variable $U = V$. Thus, the inverse solutions of

$$t = \frac{z}{\sqrt{v/k}}$$

and

$$u = v$$

are

$$z = t\sqrt{\frac{u}{k}}$$

and

$$v = u$$

The Jacobian is

$$J = \begin{vmatrix} \sqrt{\frac{u}{k}} & \frac{t}{2uk} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{u}{k}}$$

*We use the fact that $(n-1)S^2/\sigma^2$ follows a chi-square distribution with $n - 1$ degrees of freedom in Section 8-4 to find a confidence interval on the variance and standard deviation of a normal distribution.

Thus,

$$|J| = \sqrt{\frac{u}{k}}$$

and the joint probability distribution of T and U is

$$f_{TU}(t, u) = \frac{\sqrt{u}}{\sqrt{2\pi k}\, 2^{k/2}\Gamma\left(\frac{k}{2}\right)}\, u^{(k/2)-1} e^{-[(u/k)t^2 + u]/2}$$

Now, since $V > 0$, we must require that $U > 0$, and since $-\infty < Z < \infty$, $-\infty < T < \infty$. On rearranging this last equation, we have

$$f_{TU}(t, u) = \frac{1}{\sqrt{2\pi k}\, 2^{k/2}\Gamma\left(\frac{k}{2}\right)}\, u^{(k-1)/2} e^{-(u/2)[(t^2/k)+1]}, \quad 0 < u < \infty, -\infty < t < \infty$$

The probability distribution of T is found by

$$\begin{aligned} f_T(t) &= \int_0^\infty f_{TU}(t, u)\, du \\ &= \frac{1}{\sqrt{2\pi k}\, 2^{k/2}\Gamma\left(\frac{k}{2}\right)} \int_0^\infty u^{(k-1)/2} e^{-(u/2)[(t^2/k)+1]}\, du \\ &= \frac{\Gamma[(k+1)/2]}{\sqrt{\pi k}\,\Gamma\left(\frac{k}{2}\right)} \frac{1}{[(t^2/k) + 1]^{(k+1)/2}}, \qquad -\infty < t < \infty \end{aligned}$$

This is the distribution given in Theorem S8-1.

The probability distribution of the random variable T was first published by W. S. Gosset in a famous 1908 paper. Gosset was employed by the Guiness Brewers in Ireland. Since his employer discouraged publication of employee research, Gosset published these results under the pseudonym "Student." As a result, this probability distribution is sometimes called the **Student *t*-distribution.**