16 Statistical Quality Control

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Understand the role of statistical tools in quality improvement
2. Understand the different types of variability, rational subgroups, and how a control chart is used to detect assignable causes
3. Understand the general form of a Shewhart control chart and how to apply zone rules (such as the Western Electric rules) and pattern analysis to detect assignable causes
4. Construct and interpret control charts for variables such as $\overline{X}$, R, S, and individuals charts
5. Construct and interpret control charts for attributes such as P and U charts
6. Calculate and interpret process capability ratios

7. **Calculate the ARL performance for a Shewhart control chart**
8. **Construct and interpret a cumulative sum control chart**
9. **Use other statistical process control problem-solving tools**

Answers for most odd numbered exercises are at the end of the book. Answers to exercises whose numbers are surrounded by a box can be accessed in the e-text by clicking on the box. Complete worked solutions to certain exercises are also available in the e-text. These are indicated in the Answers to Selected Exercises section by a box around the exercise number. Exercises are also available for some of the text sections that appear on CD only. These exercises may be found within the e-Text immediately following the section they accompany.

16-1 QUALITY IMPROVEMENT AND STATISTICS

The quality of products and services has become a major decision factor in most businesses today. Regardless of whether the consumer is an individual, a corporation, a military defense program, or a retail store, when the consumer is making purchase decisions, he or she is likely to consider quality of equal importance to cost and schedule. Consequently, **quality improvement** has become a major concern to many U.S. corporations. This chapter is about **statistical quality control,** a collection of tools that are essential in quality-improvement activities.

Quality means **fitness for use.** For example, you or I may purchase automobiles that we expect to be free of manufacturing defects and that should provide reliable and economical transportation, a retailer buys finished goods with the expectation that they are properly packaged and arranged for easy storage and display, or a manufacturer buys raw material and expects to process it with no rework or scrap. In other words, all consumers expect that the products and services they buy will meet their requirements. Those requirements define fitness for use.

Quality or fitness for use is determined through the interaction of **quality of design** and **quality of conformance.** By quality of design we mean the different grades or levels of performance, reliability, serviceability, and function that are the result of deliberate engineering and management decisions. By quality of conformance, we mean the systematic **reduction of variability** and **elimination of defects** until every unit produced is identical and defect-free.

Some confusion exists in our society about quality improvement; some people still think that it means gold-plating a product or spending more money to develop a product or process. This thinking is wrong. Quality improvement means the systematic **elimination of waste.** Examples of waste include scrap and rework in manufacturing, inspection and testing, errors on documents (such as engineering drawings, checks, purchase orders, and plans), customer complaint hotlines, warranty costs, and the time required to do things over again that could have been done right the first time. A successful quality-improvement effort can eliminate much of this waste and lead to lower costs, higher productivity, increased customer satisfaction, increased business reputation, higher market share, and ultimately higher profits for the company.

Statistical methods play a vital role in quality improvement. Some applications are outlined below:

1. In product design and development, statistical methods, including designed experiments, can be used to compare different materials, components, or ingredients, and to help determine both system and component tolerances. This application can significantly lower development costs and reduce development time.

2. Statistical methods can be used to determine the capability of a manufacturing process. Statistical process control can be used to systematically improve a process by reducing variability.
3. Experimental design methods can be used to investigate improvements in the process. These improvements can lead to higher yields and lower manufacturing costs.
4. Life testing provides reliability and other performance data about the product. This can lead to new and improved designs and products that have longer useful lives and lower operating and maintenance costs.

Some of these applications have been illustrated in earlier chapters of this book. It is essential that engineers, scientists, and managers have an in-depth understanding of these statistical tools in any industry or business that wants to be a high-quality, low-cost producer. In this chapter we provide an introduction to the basic methods of statistical quality control that, along with experimental design, form the basis of a successful quality-improvement effort.

16-2 STATISTICAL QUALITY CONTROL

The field of statistical quality control can be broadly defined as those statistical and engineering methods that are used in measuring, monitoring, controlling, and improving quality. Statistical quality control is a field that dates back to the 1920s. Dr. Walter A. Shewhart of the Bell Telephone Laboratories was one of the early pioneers of the field. In 1924 he wrote a memorandum showing a modern control chart, one of the basic tools of statistical process control. Harold F. Dodge and Harry G. Romig, two other Bell System employees, provided much of the leadership in the development of statistically based sampling and inspection methods. The work of these three men forms much of the basis of the modern field of statistical quality control. World War II saw the widespread introduction of these methods to U.S. industry. Dr. W. Edwards Deming and Dr. Joseph M. Juran have been instrumental in spreading statistical quality-control methods since World War II.

The Japanese have been particularly successful in deploying statistical quality-control methods and have used statistical methods to gain significant advantage over their competitors. In the 1970s American industry suffered extensively from Japanese (and other foreign) competition; that has led, in turn, to renewed interest in statistical quality-control methods in the United States. Much of this interest focuses on *statistical process control* and *experimental design*. Many U.S. companies have begun extensive programs to implement these methods in their manufacturing, engineering, and other business organizations.

16-3 STATISTICAL PROCESS CONTROL

It is impractical to inspect quality into a product; the product must be built right the first time. The manufacturing process must therefore be stable or repeatable and capable of operating with little variability around the target or nominal dimension. Online statistical process control is a powerful tool for achieving process stability and improving capability through the reduction of variability.

It is customary to think of **statistical process control (SPC)** as a set of problem-solving tools that may be applied to any process. The major tools of SPC* are

1. Histogram
2. Pareto chart
3. Cause-and-effect diagram
4. Defect-concentration diagram
5. Control chart
6. Scatter diagram
7. Check sheet

Although these tools are an important part of SPC, they comprise only the technical aspect of the subject. An equally important element of SPC is attitude—a desire of all individuals in the organization for continuous improvement in quality and productivity through the systematic reduction of variability. The control chart is the most powerful of the SPC tools.

16-4 INTRODUCTION TO CONTROL CHARTS

16-4.1 Basic Principles

In any production process, regardless of how well-designed or carefully maintained it is, a certain amount of inherent or natural variability will always exist. This natural variability or "background noise" is the cumulative effect of many small, essentially unavoidable causes. When the background noise in a process is relatively small, we usually consider it an acceptable level of process performance. In the framework of statistical quality control, this natural variability is often called a "stable system of chance causes." A process that is operating with only **chance causes** of variation present is said to be in statistical control. In other words, the chance causes are an inherent part of the process.

Other kinds of variability may occasionally be present in the output of a process. This variability in key quality characteristics usually arises from three sources: improperly adjusted machines, operator errors, or defective raw materials. Such variability is generally large when compared to the background noise, and it usually represents an unacceptable level of process performance. We refer to these sources of variability that are not part of the chance cause pattern as **assignable causes.** A process that is operating in the presence of assignable causes is said to be out of control.†

Production processes will often operate in the in-control state, producing acceptable product for relatively long periods of time. Occasionally, however, assignable causes will occur, seemingly at random, resulting in a "shift" to an out-of-control state where a large proportion of the process output does not conform to requirements. A major objective of statistical process control is to quickly detect the occurrence of assignable causes or process shifts so that investigation of the process and corrective action may be undertaken before many

* Some prefer to include the experimental design methods discussed previously as part of the SPC toolkit. We did not do so, because we think of SPC as an online approach to quality improvement using techniques founded on passive observation of the process, while design of experiments is an active approach in which deliberate changes are made to the process variables. As such, designed experiments are often referred to as offline quality control.

† The terminology *chance* and *assignable* causes was developed by Dr. Walter A. Shewhart. Today, some writers use *common* cause instead of *chance* cause and *special* cause instead of *assignable* cause.

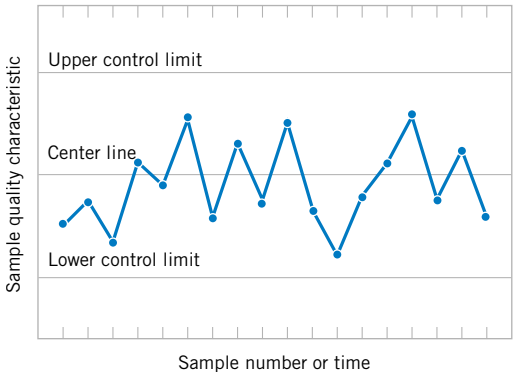

Figure 16-1 A typical control chart.

nonconforming units are manufactured. The control chart is an online process-monitoring technique widely used for this purpose.

Recall the following from Chapter 1. Figure 1-10 illustrates that adjustments to common causes of variation increase the variation of a process whereas Fig. 1-11 illustrates that actions should be taken in response to assignable causes of variation. Control charts may also be used to estimate the parameters of a production process and, through this information, to determine the capability of a process to meet specifications. The control chart can also provide information that is useful in improving the process. Finally, remember that the eventual goal of statistical process control is the *elimination of variability in the process*. Although it may not be possible to eliminate variability completely, the control chart helps reduce it as much as possible.

A typical control chart is shown in Fig. 16-1, which is a graphical display of a quality characteristic that has been measured or computed from a sample versus the sample number or time. Often, the samples are selected at periodic intervals such as every hour. The chart contains a center line (CL) that represents the average value of the quality characteristic corresponding to the in-control state. (That is, only chance causes are present.) Two other horizontal lines, called the upper control limit (UCL) and the lower control limit (LCL), are also shown on the chart. These control limits are chosen so that if the process is in control, nearly all of the sample points will fall between them. In general, as long as the points plot within the control limits, the process is assumed to be in control, and no action is necessary. However, a point that plots outside of the control limits is interpreted as evidence that the process is out of control, and investigation and corrective action are required to find and eliminate the assignable cause or causes responsible for this behavior. The sample points on the control chart are usually connected with straight-line segments so that it is easier to visualize how the sequence of points has evolved over time.

Even if all the points plot inside the control limits, if they behave in a systematic or nonrandom manner, this is an indication that the process is out of control. For example, if 18 of the last 20 points plotted above the center line but below the upper control limit and only two of these points plotted below the center line but above the lower control limit, we would be very suspicious that something was wrong. If the process is in control, all the plotted points should have an essentially random pattern. Methods designed to find sequences or nonrandom patterns can be applied to control charts as an aid in detecting out-of-control conditions. A particular nonrandom pattern usually appears on a control chart for a reason, and if that reason can be found and eliminated, process performance can be improved.

There is a close connection between control charts and hypothesis testing. Essentially, the control chart is a test of the hypothesis that the process is in a state of statistical control. A point plotting within the control limits is equivalent to failing to reject the hypothesis of statistical control, and a point plotting outside the control limits is equivalent to rejecting the hypothesis of statistical control.

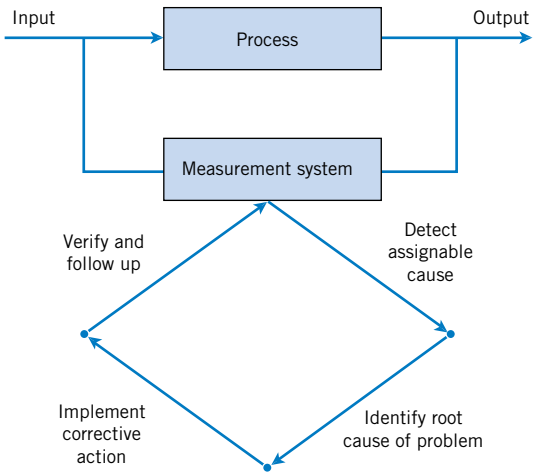

Figure 16-2 Process improvement using the control chart.

We give a general *model* for a control chart. Let W be a sample statistic that measures some quality characteristic of interest, and suppose that the mean of W is μ_W and the standard deviation of W is σ_W.* Then the center line, the upper control limit, and the lower control limit become

$$\begin{aligned} UCL &= \mu_W + k\sigma_W \\ CL &= \mu_W \\ LCL &= \mu_W - k\sigma_W \end{aligned} \tag{16-1}$$

where k is the "distance" of the control limits from the center line, expressed in standard deviation units. A common choice is $k = 3$. This general theory of control charts was first proposed by Dr. Walter A. Shewhart, and control charts developed according to these principles are often called **Shewhart control charts.**

The control chart is a device for describing exactly what is meant by statistical control; as such, it may be used in a variety of ways. In many applications, it is used for online process monitoring. That is, sample data are collected and used to construct the control chart, and if the sample values of $\bar{x}$ (say) fall within the control limits and do not exhibit any systematic pattern, we say the process is in control at the level indicated by the chart. Note that we may be interested here in determining *both* whether the past data came from a process that was in control and whether future samples from this process indicate statistical control.

The most important use of a control chart is to *improve* the process. We have found that, generally

1. Most processes do not operate in a state of statistical control.
2. Consequently, the routine and attentive use of control charts will identify assignable causes. If these causes can be eliminated from the process, variability will be reduced and the process will be improved.

This process-improvement activity using the control chart is illustrated in Fig. 16-2. Notice that:

* Note that "sigma" refers to the standard deviation of the statistic plotted on the chart (i.e., σ_W), not the standard deviation of the quality characteristic.

3. The control chart will only *detect* assignable causes. Management, operator, and engineering *action* will usually be necessary to eliminate the assignable cause. An action plan for responding to control chart signals is vital.

In identifying and eliminating assignable causes, it is important to find the underlying **root cause** of the problem and to attack it. A cosmetic solution will not result in any real, long-term process improvement. Developing an effective system for corrective action is an essential component of an effective SPC implementation.

We may also use the control chart as an *estimating device*. That is, from a control chart that exhibits statistical control, we may estimate certain process parameters, such as the mean, standard deviation, and fraction nonconforming or fallout. These estimates may then be used to determine the *capability* of the process to produce acceptable products. Such **process capability studies** have considerable impact on many management decision problems that occur over the product cycle, including make-or-buy decisions, plant and process improvements that reduce process variability, and contractual agreements with customers or suppliers regarding product quality.

Control charts may be classified into two general types. Many quality characteristics can be measured and expressed as numbers on some continuous scale of measurement. In such cases, it is convenient to describe the quality characteristic with a measure of central tendency and a measure of variability. Control charts for central tendency and variability are collectively called **variables control charts.** The $\overline{X}$ chart is the most widely used chart for monitoring central tendency, whereas charts based on either the sample range or the sample standard deviation are used to control process variability. Many quality characteristics are not measured on a continuous scale or even a quantitative scale. In these cases, we may judge each unit of product as either conforming or nonconforming on the basis of whether or not it possesses certain attributes, or we may count the number of nonconformities (defects) appearing on a unit of product. Control charts for such quality characteristics are called **attributes control charts.**

Control charts have had a long history of use in industry. There are at least five reasons for their popularity:

1. **Control charts are a proven technique for improving productivity.** A successful control chart program will reduce scrap and rework, which are the primary productivity killers in *any* operation. If you reduce scrap and rework, productivity increases, cost decreases, and production capacity (measured in the number of *good* parts per hour) increases.
2. **Control charts are effective in defect prevention.** The control chart helps keep the process in control, which is consistent with the "do it right the first time" philosophy. It is never cheaper to sort out the "good" units from the "bad" later on than it is to build them correctly initially. If you do not have effective process control, you are paying someone to make a nonconforming product.
3. **Control charts prevent unnecessary process adjustments.** A control chart can distinguish between background noise and abnormal variation; no other device, including a human operator, is as effective in making this distinction. If process operators adjust the process based on periodic tests unrelated to a control chart program, they will often overreact to the background noise and make unneeded adjustments. These unnecessary adjustments can result in a deterioration of process performance. In other words, the control chart is consistent with the "if it isn't broken, don't fix it" philosophy.
4. **Control charts provide diagnostic information.** Frequently, the pattern of points on the control chart will contain information that is of diagnostic value to an

experienced operator or engineer. This information allows the operator to implement a change in the process that will improve its performance.

5. **Control charts provide information about process capability.** The control chart provides information about the value of important process parameters and their stability over time. This allows an estimate of process capability to be made. This information is of tremendous use to product and process designers.

Control charts are among the most effective management control tools, and they are as important as cost controls and material controls. Modern computer technology has made it easy to implement control charts in any type of process, because data collection and analysis can be performed on a microcomputer or a local area network terminal in realtime, online at the work center.

16-4.2 Design of a Control Chart

To illustrate these ideas, we give a simplified example of a control chart. In manufacturing automobile engine piston rings, the inside diameter of the rings is a critical quality characteristic. The process mean inside ring diameter is 74 millimeters, and it is known that the standard deviation of ring diameter is 0.01 millimeters. A control chart for average ring diameter is shown in Fig. 16-3. Every hour a random sample of five rings is taken, the average ring diameter of the sample (say $\bar{x}$) is computed, and $\bar{x}$ is plotted on the chart. Because this control chart utilizes the sample mean $\overline{X}$ to monitor the process mean, it is usually called an $\overline{X}$ control chart. Note that all the points fall within the control limits, so the chart indicates that the process is in statistical control.

Consider how the control limits were determined. The process average is 74 millimeters, and the process standard deviation is $\sigma = 0.01$ millimeters. Now if samples of size $n = 5$ are taken, the standard deviation of the sample average $\overline{X}$ is

$$\sigma_{\overline{X}} = \frac{\sigma}{\sqrt{n}} = \frac{0.01}{\sqrt{5}} = 0.0045$$

Therefore, if the process is in control with a mean diameter of 74 millimeters, by using the central limit theorem to assume that $\overline{X}$ is approximately normally distributed, we would expect approximately $100(1 - \alpha)\%$ of the sample mean diameters $\overline{X}$ to fall between $74 + z_{\alpha/2}(0.0045)$ and $74 - z_{\alpha/2}(0.0045)$. As discussed above, we customarily choose the

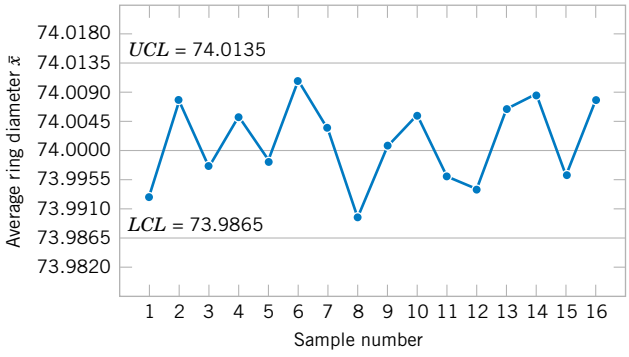

Figure 16-3 $\overline{X}$ control chart for piston ring diameter.

constant $z_{\alpha/2}$ to be 3, so the upper and lower control limits become

$$UCL = 74 + 3(0.0045) = 74.0135$$

and

$$LCL = 74 - 3(0.0045) = 73.9865$$

as shown on the control chart. These are the 3-sigma control limits referred to above. Note that the use of 3-sigma limits implies that $\alpha = 0.0027$; that is, the probability that the point plots outside the control limits when the process is in control is 0.0027. The width of the control limits is inversely related to the sample size n for a given multiple of sigma. Choosing the control limits is equivalent to setting up the critical region for testing the hypothesis

$$H_0: \mu = 74$$
$$H_1: \mu \neq 74$$

where $\sigma = 0.01$ is known. Essentially, the control chart tests this hypothesis repeatedly at different points in time.

In designing a control chart, we must specify both the sample size to use and the frequency of sampling. In general, larger samples will make it easier to detect small shifts in the process. When choosing the sample size, we must keep in mind the size of the shift that we are trying to detect. If we are interested in detecting a relatively large process shift, we use smaller sample sizes than those that would be employed if the shift of interest were relatively small.

We must also determine the frequency of sampling. The most desirable situation from the point of view of detecting shifts would be to take large samples very frequently; however, this is usually not economically feasible. The general problem is one of *allocating sampling effort*. That is, either we take small samples at short intervals or larger samples at longer intervals. Current industry practice tends to favor smaller, more frequent samples, particularly in high-volume manufacturing processes or where a great many types of assignable causes can occur. Furthermore, as automatic sensing and measurement technology develops, it is becoming possible to greatly increase frequencies. Ultimately, every unit can be tested as it is manufactured. This capability will not eliminate the need for control charts because the test system will not prevent defects. The increased data will increase the effectiveness of process control and improve quality.

16-4.3 Rational Subgroups

A fundamental idea in the use of control charts is to collect sample data according to what Shewhart called the **rational subgroup** concept. Generally, this means that subgroups or samples should be selected so that to the extent possible, the variability of the observations within a subgroup should include all the chance or natural variability and exclude the assignable variability. Then, the control limits will represent bounds for all the chance variability and not the assignable variability. Consequently, assignable causes will tend to generate points that are outside of the control limits, while chance variability will tend to generate points that are within the control limits.

When control charts are applied to production processes, the time order of production is a logical basis for rational subgrouping. Even though time order is preserved, it is still possible to form subgroups erroneously. If some of the observations in the subgroup are taken at the end of one 8-hour shift and the remaining observations are taken at the start of the next 8-hour shift,

any differences between shifts might not be detected. Time order is frequently a good basis for forming subgroups because it allows us to detect assignable causes that occur over time.

Two general approaches to constructing rational subgroups are used. In the first approach, each subgroup consists of units that were produced at the same time (or as closely together as possible). This approach is used when the primary purpose of the control chart is to detect process shifts. It minimizes variability due to assignable causes *within* a sample, and it maximizes variability *between* samples if assignable causes are present. It also provides better estimates of the standard deviation of the process in the case of variables control charts. This approach to rational subgrouping essentially gives a "snapshot" of the process at each point in time where a sample is collected.

In the second approach, each sample consists of units of product that are representative of *all* units that have been produced since the last sample was taken. Essentially, each subgroup is a *random sample* of *all* process output over the sampling interval. This method of rational subgrouping is often used when the control chart is employed to make decisions about the acceptance of all units of product that have been produced since the last sample. In fact, if the process shifts to an out-of-control state and then back in control again *between* samples, it is sometimes argued that the first method of rational subgrouping defined above will be ineffective against these types of shifts, and so the second method must be used.

When the rational subgroup is a random sample of all units produced over the sampling interval, considerable care must be taken in interpreting the control charts. If the process mean drifts between several levels during the interval between samples, the range of observations within the sample may consequently be relatively large. It is the within-sample variability that determines the width of the control limits on an $\overline{X}$ chart, so this practice will result in wider limits on the $\overline{X}$ chart. This makes it harder to detect shifts in the mean. In fact, we can often make *any* process appear to be in statistical control just by stretching out the interval between observations in the sample. It is also possible for shifts in the process average to cause points on a control chart for the range or standard deviation to plot out of control, even though no shift in process variability has taken place.

There are other bases for forming rational subgroups. For example, suppose a process consists of several machines that pool their output into a common stream. If we sample from this common stream of output, it will be very difficult to detect whether or not some of the machines are out of control. A logical approach to rational subgrouping here is to apply control chart techniques to the output for each individual machine. Sometimes this concept needs to be applied to different heads on the same machine, different workstations, different operators, and so forth.

The rational subgroup concept is very important. The proper selection of samples requires careful consideration of the process, with the objective of obtaining as much useful information as possible from the control chart analysis.

16-4.4 Analysis of Patterns on Control Charts

A control chart may indicate an out-of-control condition either when one or more points fall beyond the control limits, or when the plotted points exhibit some nonrandom pattern of behavior. For example, consider the $\overline{X}$ chart shown in Fig. 16-4. Although all 25 points fall within the control limits, the points do not indicate statistical control because their pattern is very nonrandom in appearance. Specifically, we note that 19 of the 25 points plot below the center line, while only 6 of them plot above. If the points are truly random, we should expect a more even distribution of them above and below the center line. We also observe that following the fourth point, five points in a row increase in magnitude. This arrangement of points is called a **run.** Since the observations are increasing, we could call it a run up; similarly, a sequence of decreasing points

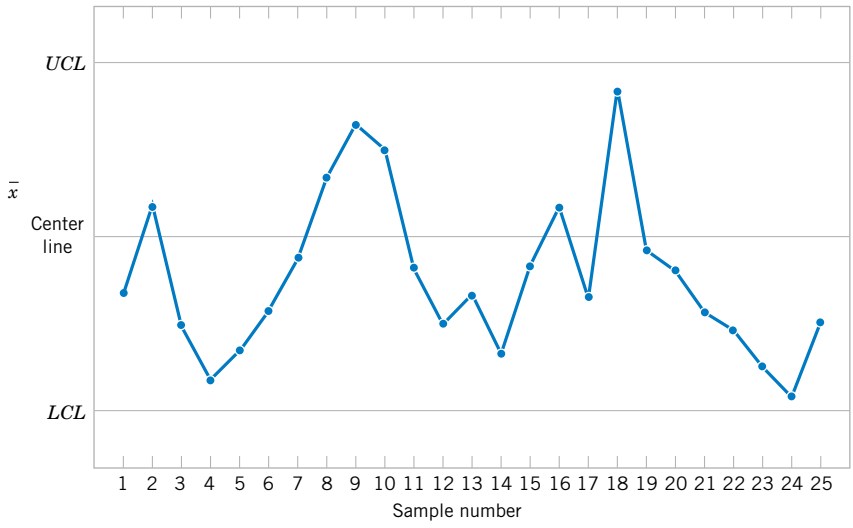

Figure 16-4 An $\overline{X}$ control chart.

is called a run down. This control chart has an unusually long run up (beginning with the fourth point) and an unusually long run down (beginning with the eighteenth point).

In general, we define a run as a sequence of observations of the same type. In addition to runs up and runs down, we could define the types of observations as those above and below the center line, respectively, so two points in a row above the center line would be a run of length 2.

A run of length 8 or more points has a very low probability of occurrence in a random sample of points. Consequently, any type of run of length 8 or more is often taken as a signal of an out-of-control condition. For example, eight consecutive points on one side of the center line will indicate that the process is out of control.

Although runs are an important measure of nonrandom behavior on a control chart, other types of patterns may also indicate an out-of-control condition. For example, consider the $\overline{X}$ chart in Fig. 16-5. Note that the plotted sample averages exhibit a cyclic behavior, yet they all fall within the control limits. Such a pattern may indicate a problem with the process, such as operator fatigue, raw material deliveries, and heat or stress buildup. The yield may be improved by eliminating or reducing the sources of variability causing this cyclic behavior (see Fig. 16-6). In Fig. 16-6, *LSL* and *USL* denote the lower and upper specification limits of the process. These limits represent bounds within which acceptable product must fall and they are often based on customer requirements.

The problem is one of **pattern recognition,** that is, recognizing systematic or nonrandom patterns on the control chart and identifying the reason for this behavior. The ability to interpret

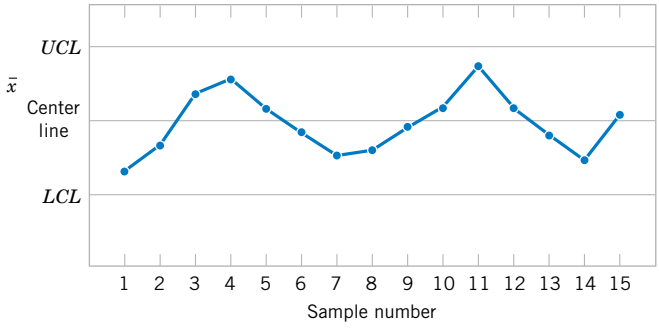

Figure 16-5 An $\overline{X}$ chart with a cyclic pattern.

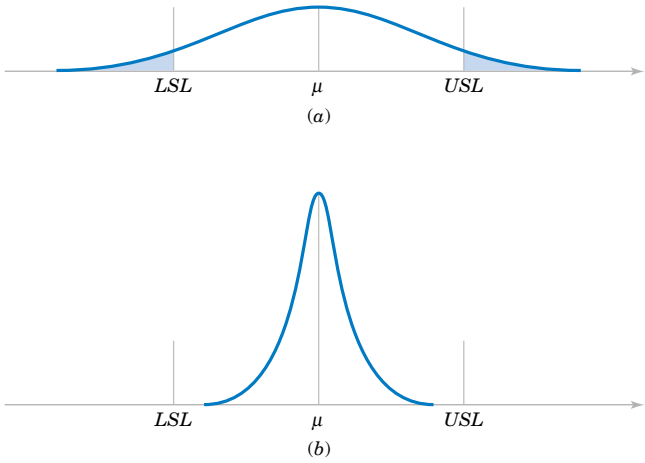

Figure 16-6 (a) Variability with the cyclic pattern. (b) Variability with the cyclic pattern eliminated.

a particular pattern in terms of assignable causes requires experience and knowledge of the process. That is, we must not only know the statistical principles of control charts, but we must also have a good understanding of the process.

The Western Electric Handbook (1956) suggests a set of decision rules for detecting nonrandom patterns on control charts. Specifically, the **Western Electric rules** would conclude that the process is out of control if either

1. One point plots outside 3-sigma control limits.
2. Two out of three consecutive points plot beyond a 2-sigma limit.
3. Four out of five consecutive points plot at a distance of 1-sigma or beyond from the center line.
4. Eight consecutive points plot on one side of the center line.

We have found these rules very effective in practice for enhancing the sensitivity of control charts. Rules 2 and 3 apply to one side of the center line at a time. That is, a point above the *upper* 2-sigma limit followed immediately by a point below the *lower* 2-sigma limit would not signal an out-of-control alarm.

Figure 16-7 shows an $\overline{X}$ control chart for the piston ring process with the 1-sigma, 2-sigma, and 3-sigma limits used in the Western Electric procedure. Notice that these inner

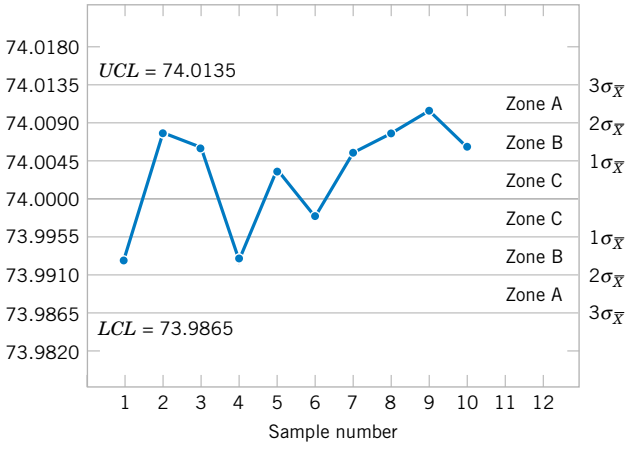

Figure 16-7 The Western Electric zone rules.

limits (sometimes called **warning limits**) partition the control chart into three zones A, B, and C on each side of the center line. Consequently, the Western Electric rules are sometimes called the **run rules** for control charts. Notice that the last four points fall in zone B or beyond. Thus, since four of five consecutive points exceed the 1-sigma limit, the Western Electric procedure will conclude that the pattern is nonrandom and the process is out of control.

16-5 $\overline{X}$ AND R OR S CONTROL CHARTS

When dealing with a quality characteristic that can be expressed as a measurement, it is customary to monitor both the mean value of the quality characteristic and its variability. Control over the average quality is exercised by the control chart for averages, usually called the $\overline{X}$ chart. Process variability can be controlled by either a range chart (R chart) or a standard deviation chart (S chart), depending on how the population standard deviation is estimated.

Suppose that the process mean and standard deviation μ and σ are known and that we can assume that the quality characteristic has a normal distribution. Consider the $\overline{X}$ chart. As discussed previously, we can use μ as the center line for the control chart, and we can place the upper and lower 3-sigma limits at

$$\begin{aligned} UCL &= \mu + 3\sigma/\sqrt{n} \\ LCL &= \mu - 3\sigma/\sqrt{n} \\ CL &= \mu \end{aligned} \tag{16-2}$$

When the parameters μ and σ are unknown, we usually estimate them on the basis of preliminary samples, taken when the process is thought to be in control. We recommend the use of at least 20 to 25 preliminary samples. Suppose m preliminary samples are available, each of size n. Typically, n will be 4, 5, or 6; these relatively small sample sizes are widely used and often arise from the construction of rational subgroups. Let the sample mean for the ith sample be $\overline{X}_i$. Then we estimate the mean of the population, μ, by the **grand mean**

$$\hat{\mu} = \overline{\overline{X}} = \frac{1}{m}\sum_{i=1}^{m}\overline{X}_i \tag{16-3}$$

Thus, we may take $\overline{\overline{X}}$ as the center line on the $\overline{X}$ control chart.

We may estimate σ from either the standard deviation or the range of the observations within each sample. The sample size is relatively small, so there is little loss in efficiency in estimating σ from the sample ranges.

The relationship between the range R of a sample from a normal population with known parameters and the standard deviation of that population is needed. Since R is a random variable, the quantity $W = R/\sigma$, called the relative range, is also a random variable. The parameters of the distribution of W have been determined for any sample size n. The mean of the distribution of W is called d_2, and a table of d_2 for various n is given in Appendix Table X.

The standard deviation of W is called d_3. Because $R = \sigma W$

$$\mu_R = d_2\sigma \qquad \sigma_R = d_3\sigma \tag{16-4}$$

Let R_i be the range of the ith sample, and let

$$\bar{R} = \frac{1}{m}\sum_{i=1}^{m} R_i \tag{16-5}$$

be the average range. Then $\bar{R}$ is an estimator of μ_R and from Equation 16-4 an unbiased estimator of σ is

$$\hat{\sigma} = \frac{\bar{R}}{d_2} \tag{16-6}$$

Therefore, we may use as our upper and lower control limits for the $\bar{X}$ chart

$$UCL = \bar{\bar{X}} + \frac{3}{d_2\sqrt{n}}\bar{R} \qquad LCL = \bar{\bar{X}} - \frac{3}{d_2\sqrt{n}}\bar{R} \tag{16-7}$$

Define the constant

$$A_2 = \frac{3}{d_2\sqrt{n}} \tag{16-8}$$

Now, once we have computed the sample values $\bar{\bar{x}}$ and $\bar{r}$, the $\bar{X}$ control chart may be defined as follows:

$\bar{X}$ Control Chart (from $\bar{R}$)

The center line and upper and lower control limits for an $\bar{X}$ control chart are

$$UCL = \bar{\bar{x}} + A_2\bar{r} \qquad CL = \bar{\bar{x}} \qquad LCL = \bar{\bar{x}} - A_2\bar{r} \tag{16-9}$$

where the constant A_2 is tabulated for various sample sizes in Appendix Table X.

The parameters of the R chart may also be easily determined. The center line will obviously be $\bar{R}$. To determine the control limits, we need an estimate of σ_R, the standard deviation of R. Once again, assuming the process is in control, the distribution of the relative range, W, will be useful. We may estimate σ_R from Equation 16-4 as

$$\hat{\sigma}_R = d_3\hat{\sigma} = d_3\frac{\bar{R}}{d_2} \tag{16-10}$$

and we would use as the upper and lower control limits on the R chart

$$UCL = \overline{R} + \frac{3d_3}{d_2}\overline{R} = \left(1 + \frac{3d_3}{d_2}\right)\overline{R}$$
$$LCL = \overline{R} - \frac{3d_3}{d_2}\overline{R} = \left(1 - \frac{3d_3}{d_2}\right)\overline{R} \qquad (16\text{-}11)$$

Setting $D_3 = 1 - 3d_3/d_2$ and $D_4 = 1 + 3d_3/d_2$ leads to the following definition.

R Chart

The center line and upper and lower control limits for an R chart are

$$UCL = D_4\bar{r} \qquad CL = \bar{r} \qquad LCL = D_3\bar{r} \qquad (16\text{-}12)$$

where $\bar{r}$ is the sample average range, and the constants D_3 and D_4 are tabulated for various sample sizes in Appendix Table X.

The LCL for an R chart can be a negative number. In that case, it is customary to set LCL to zero. Because the points plotted on an R chart are nonnegative, no points can fall below an LCL of zero.

When preliminary samples are used to construct limits for control charts, these limits are customarily treated as trial values. Therefore, the m sample means and ranges should be plotted on the appropriate charts, and any points that exceed the control limits should be investigated. If assignable causes for these points are discovered, they should be eliminated and new limits for the control charts determined. In this way, the process may be eventually brought into statistical control and its inherent capabilities assessed. Other changes in process centering and dispersion may then be contemplated. Also, we often study the R chart first because if the process variability is not constant over time the control limits calculated for the $\overline{X}$ chart can be misleading.

Rather than base control charts on ranges, a more modern approach is to calculate the standard deviation of each subgroup and plot these standard deviations to monitor the process standard deviation σ. This is called an S chart. When an S chart is used, it is common to use these standard deviations to develop control limits for the $\overline{X}$ chart. Typically, the sample size used for subgroups is small (fewer than 10) and in that case there is usually little difference in the $\overline{X}$ chart generated from ranges or standard deviations. However, because computer software is often used to implement control charts, S charts are quite common. Details to construct these charts follow.

In Section 7-2.2 on the CD, it was shown that S is a biased estimator of σ. That is, $E(S) = c_4\sigma$ where c_4 is a constant that is near, but not equal to, 1. Furthermore, a calculation similar to the one used for $E(S)$ can derive the standard deviation of the statistic S with the result $\sigma\sqrt{1 - c_4^2}$. Therefore, the center line and three-sigma control limits for S are

$$LCL = c_4\sigma - 3\sigma\sqrt{1 - c_4^2} \qquad CL = c_4\sigma$$
$$UCL = c_4\sigma + 3\sigma\sqrt{1 - c_4^2} \qquad (16\text{-}13)$$

Assume that there are m preliminary samples available, each of size n, and let S_i denote the standard deviation of the ith sample. Define

$$\bar{S} = \frac{1}{m}\sum_{i=1}^{m} S_i \tag{16-14}$$

Because $E(\bar{S}) = c_4\sigma$, an unbiased estimator of σ is $\bar{S}/c_4$ That is,

$$\hat{\sigma} = \bar{S}/c_4 \tag{16-15}$$

A control chart for standard deviations follows.

S Chart

$$UCL = \bar{s} + 3\frac{\bar{s}}{c_4}\sqrt{1 - c_4^2} \qquad CL = \bar{s} \qquad LCL = \bar{s} - 3\frac{\bar{s}}{c_4}\sqrt{1 - c_4^2} \tag{16-16}$$

The LCL for an S chart can be a negative number, in that case, it is customary to set LCL to zero. When an S chart is used, the estimate for σ in Equation 16-15 is commonly used to calculate the control limits for an $\bar{X}$ chart. This produces the following control limits for an $\bar{X}$ chart.

$\bar{X}$ Control Chart (from $\bar{S}$)

$$UCL = \bar{\bar{x}} + 3\frac{\bar{s}}{c_4\sqrt{n}} \qquad CL = \bar{\bar{x}} \qquad LCL = \bar{s} - 3\frac{\bar{s}}{c_4\sqrt{n}} \tag{16-17}$$

EXAMPLE 16-1

A component part for a jet aircraft engine is manufactured by an investment casting process. The vane opening on this casting is an important functional parameter of the part. We will illustrate the use of $\bar{X}$ and R control charts to assess the statistical stability of this process. Table 16-1 presents 20 samples of five parts each. The values given in the table have been coded by using the last three digits of the dimension; that is, 31.6 should be 0.50316 inch.

The quantities $\bar{\bar{x}} = 33.3$ and $\bar{r} = 5.8$ are shown at the foot of Table 16-1. The value of A_2 for samples of size 5 is $A_2 = 0.577$. Then the trial control limits for the $\bar{X}$ chart are

$$\bar{x} \pm A_2\bar{r} = 33.32 \pm (0.577)(5.8) = 33.32 \pm 3.35$$

or

$$UCL = 36.67 \qquad LCL = 29.97$$

Table 16-1 Vane-Opening Measurements

Sample Number	x_1	x_2	x_3	x_4	x_5	$\bar{x}$	r	s
1	33	29	31	32	33	31.6	4	1.67332
2	33	31	35	37	31	33.4	6	2.60768
3	35	37	33	34	36	35.0	4	1.58114
4	30	31	33	34	33	32.2	4	1.64317
5	33	34	35	33	34	33.8	2	0.83666
6	38	37	39	40	38	38.4	3	1.14018
7	30	31	32	34	31	31.6	4	1.51658
8	29	39	38	39	39	36.8	10	4.38178
9	28	33	35	36	43	35.0	15	5.43139
10	38	33	32	35	32	34.0	6	2.54951
11	28	30	28	32	31	29.8	4	1.78885
12	31	35	35	35	34	34.0	4	1.73205
13	27	32	34	35	37	33.0	10	3.80789
14	33	33	35	37	36	34.8	4	1.78885
15	35	37	32	35	39	35.6	7	2.60768
16	33	33	27	31	30	30.8	6	2.48998
17	35	34	34	30	32	33.0	5	2.00000
18	32	33	30	30	33	31.6	3	1.51658
19	25	27	34	27	28	28.2	9	3.42053
20	35	35	36	33	30	33.8	6	2.38747
						$\bar{\bar{x}} = 33.32$	$\bar{r} = 5.8$	$\bar{s} = 2.345$

For the R chart, the trial control limits are

$$UCL = D_4\bar{r} = (2.115)(5.8) = 12.27$$
$$LCL = D_3\bar{r} = (0)(5.8) = 0$$

The $\overline{X}$ and R control charts with these trial control limits are shown in Fig. 16-8. Notice that samples 6, 8, 11, and 19 are out of control on the $\overline{X}$ chart and that sample 9 is out of control on the R chart. (These points are labeled with a "1" because they violate the first Western Electric rule.)

For the S chart, the value of $c_4 = 0.94$.
Therefore,

$$\frac{3\bar{s}}{c_4}\sqrt{1 - c_4^2} = \frac{3(2.345)}{0.94}\sqrt{1 - 0.94^2} = 2.553$$

and the trial control limits are

$$UCL = 2.345 + 2.553 = 4.898$$
$$LCL = 2.345 - 2.553 = -0.208$$

The LCL is set to zero. If $\bar{s}$ is used to determine the control limits for the $\overline{X}$ chart,

$$\bar{\bar{x}} \pm \frac{3\bar{s}}{c_4\sqrt{n}} = 33.32 \pm \frac{3(2.345)}{0.94} = 33.32 \pm 3.35$$

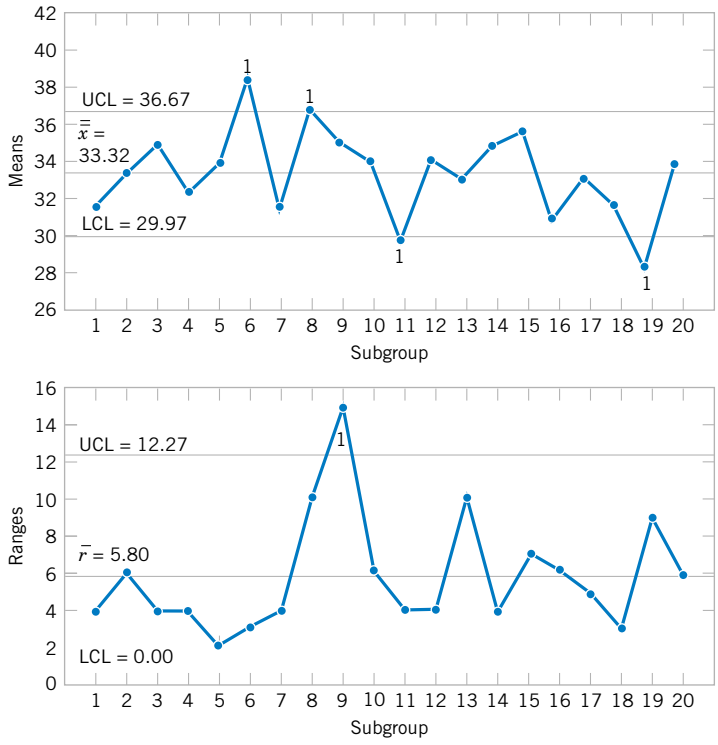

Figure 16-8 The $\overline{X}$ and R control charts for vane opening.

and this result is nearly the same as from $\bar{r}$. The S chart is shown in Fig. 16-9. Because the control limits for the $\overline{X}$ chart calculated from $\bar{s}$ are nearly the same as from $\bar{r}$, the chart is not shown.

Suppose that all of these assignable causes can be traced to a defective tool in the wax-molding area. We should discard these five samples and recompute the limits for the $\overline{X}$ and R charts. These new revised limits are, for the $\overline{X}$ chart,

$$UCL = \bar{\bar{x}} + A_2\bar{r} = 33.21 + (0.577)(5.0) = 36.10$$

$$LCL = \bar{\bar{x}} - A_2\bar{r} = 33.21 - (0.577)(5.0) = 30.33$$

and for the R chart,

$$UCL = D_4\bar{r} = (2.115)(5.0) = 10.57$$

$$LCL = D_3\bar{r} = (0)(5.0) = 0$$

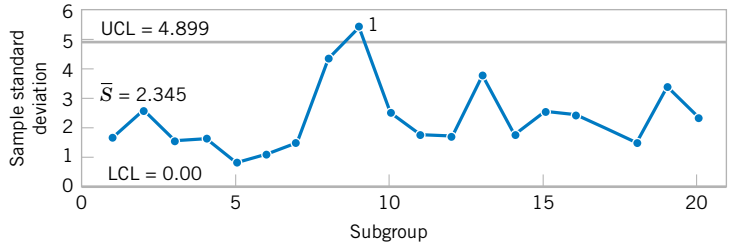

Figure 16-9. The S control chart for vane opening.

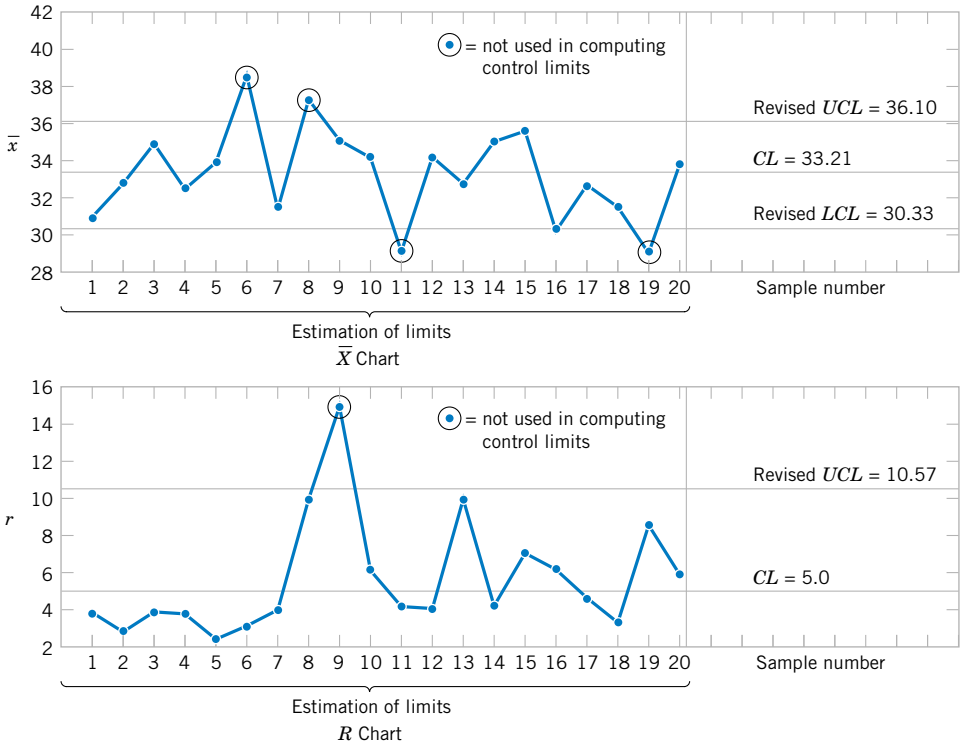

Figure 16-10 The $\overline{X}$ and R control charts for vane opening, revised limits.

The revised control charts are shown in Fig. 16-10. Notice that we have treated the first 20 preliminary samples as **estimation data** with which to establish control limits. These limits can now be used to judge the statistical control of future production. As each new sample becomes available, the values of $\bar{x}$ and r should be computed and plotted on the control charts. It may be desirable to revise the limits periodically, even if the process remains stable. The limits should always be revised when process improvements are made.

Computer Construction of $\overline{X}$ and R Control Charts

Many computer programs construct $\overline{X}$ and R control charts. Figures 16-8 and 16-10 show charts similar to those produced by Minitab for the vane-opening data. This program will allow the user to select any multiple of sigma as the width of the control limits and use the Western Electric rules to detect out-of-control points. The program will also prepare a summary report as in Table 16-2 and exclude subgroups from the calculation of the control limits.

Table 16-2 Summary Report from Minitab for the Vane-Opening Data

Test Results for Xbar Chart
TEST 1. One point more than 3.00 sigmas from center line.
Test Failed at points: 6 8 11 19

Test Results for R Chart
TEST 1. One point more than 3.00 sigmas from center line.
Test Failed at points: 9

EXERCISES FOR SECTION 16-5

16-1. An extrusion die is used to produce aluminum rods. The diameter of the rods is a critical quality characteristic. The following table shows $\bar{x}$ and r values for 20 samples of five rods each. Specifications on the rods are 0.5035 ± 0.0010 inch. The values given are the last three digits of the measurement; that is, 34.2 is read as 0.50342.

Sample	$\bar{x}$	r
1	34.2	3
2	31.6	4
3	31.8	4
4	33.4	5
5	35.0	4
6	32.1	2
7	32.6	7
8	33.8	9
9	34.8	10
10	38.6	4
11	35.4	8
12	34.0	6
13	36.0	4
14	37.2	7
15	35.2	3
16	33.4	10
17	35.0	4
18	34.4	7
19	33.9	8
20	34.0	4

(a) Using all the data, find trial control limits for $\overline{X}$ and R charts, construct the chart, and plot the data.

(b) Use the trial control limits from part (a) to identify out-of-control points. If necessary, revise your control limits, assuming that any samples that plot outside the control limits can be eliminated.

16-2. Twenty-five samples of size 5 are drawn from a process at one-hour intervals, and the following data are obtained:

$$\sum_{i=1}^{25} \bar{x}_i = 362.75 \qquad \sum_{i=1}^{25} r_i = 8.60 \qquad \sum_{i=1}^{25} s_i = 3.64$$

(a) Find trial control limits for $\overline{X}$ and R charts.

(b) Repeat part (a) for $\overline{X}$ and S charts.

16-3. The pull strength of a wire-bonded lead for an integrated circuit monitored. The following table provides data for 20 samples each of size three.

(a) Use all the data to determine trial control limits for $\overline{X}$ and R charts, construct the control limits, and plot the data.

(b) Use the control limits from part (a) to identify out-of-control points. If necessary, revise your control limits assuming that any samples that plot outside of the control limits can be eliminated.

(c) Repeat parts (a) and (b) for $\overline{X}$ and S charts.

Sample Number	x_1	x_2	x_3
1	15.4	15.6	15.3
2	15.4	17.1	15.2
3	16.1	16.1	13.5
4	13.5	12.5	10.2
5	18.3	16.1	17.0
6	19.2	17.2	19.4
7	14.1	12.4	11.7
8	15.6	13.3	13.6
9	13.9	14.9	15.5
10	18.7	21.2	20.1
11	15.3	13.1	13.7
12	16.6	18.0	18.0
13	17.0	15.2	18.1
14	16.3	16.5	17.7
15	8.4	7.7	8.4
16	11.1	13.8	11.9
17	16.5	17.1	18.5
18	18.0	14.1	15.9
19	17.8	17.3	12.0
20	11.5	10.8	11.2

16-4. Samples of size $n = 6$ are collected from a process every hour. After 20 samples have been collected, we calculate $\bar{\bar{x}} = 20.0$ and $\bar{r}/d_2 = 1.4$.

(a) Find trial control limits for $\overline{X}$ and R charts.

(b) If $\bar{s}/c_4 = 1.5$, determine trial control limits for $\overline{X}$ and S charts.

16-5. Control charts for $\overline{X}$ and R are to be set up for an important quality characteristic. The sample size is $n = 5$, and $\bar{x}$ and r are computed for each of 35 preliminary samples. The summary data are

$$\sum_{i=1}^{35} \bar{x}_i = 7805 \qquad \sum_{i=1}^{35} r_i = 1200$$

(a) Find trial control limits for $\overline{X}$ and R charts.

(b) Assuming that the process is in control, estimate the process mean and standard deviation.

16-6. Control charts are to be constructed for samples of size $n = 4$, and $\bar{x}$ and s are computed for each of 20 preliminary samples as follows:

$$\sum_{i=1}^{20} \bar{x}_i = 4460 \qquad \sum_{i=1}^{20} s_i = 271.6$$

(a) Determine trial control limits for $\overline{X}$ and S charts.
(b) Assuming the process is in control, estimate the process mean and standard deviation.

16-7. The thickness of a metal part is an important quality parameter. Data on thickness (in inches) are given in the following table, for 25 samples of five parts each.

Sample Number	x_1	x_2	x_3	x_4	x_5
1	0.0629	0.0636	0.0640	0.0635	0.0640
2	0.0630	0.0631	0.0622	0.0625	0.0627
3	0.0628	0.0631	0.0633	0.0633	0.0630
4	0.0634	0.0630	0.0631	0.0632	0.0633
5	0.0619	0.0628	0.0630	0.0619	0.0625
6	0.0613	0.0629	0.0634	0.0625	0.0628
7	0.0630	0.0639	0.0625	0.0629	0.0627
8	0.0628	0.0627	0.0622	0.0625	0.0627
9	0.0623	0.0626	0.0633	0.0630	0.0624
10	0.0631	0.0631	0.0633	0.0631	0.0630
11	0.0635	0.0630	0.0638	0.0635	0.0633
12	0.0623	0.0630	0.0630	0.0627	0.0629
13	0.0635	0.0631	0.0630	0.0630	0.0630
14	0.0645	0.0640	0.0631	0.0640	0.0642
15	0.0619	0.0644	0.0632	0.0622	0.0635
16	0.0631	0.0627	0.0630	0.0628	0.0629
17	0.0616	0.0623	0.0631	0.0620	0.0625
18	0.0630	0.0630	0.0626	0.0629	0.0628
19	0.0636	0.0631	0.0629	0.0635	0.0634
20	0.0640	0.0635	0.0629	0.0635	0.0634
21	0.0628	0.0625	0.0616	0.0620	0.0623
22	0.0615	0.0625	0.0619	0.0619	0.0622
23	0.0630	0.0632	0.0630	0.0631	0.0630
24	0.0635	0.0629	0.0635	0.0631	0.0633
25	0.0623	0.0629	0.0630	0.0626	0.0628

(a) Using all the data, find trial control limits for $\overline{X}$ and R charts, construct the chart, and plot the data. Is the process in statistical control?
(b) Repeat part (a) for $\overline{X}$ and S charts.
(c) Use the trial control limits from part (a) to identify out-of-control points. List the sample numbers of the out-of-control points.

16-8. The copper content of a plating bath is measured three times per day, and the results are reported in ppm. The $\bar{x}$ and r values for 25 days are shown in the following table:

(a) Using all the data, find trial control limits for $\overline{X}$ and R charts, construct the chart, and plot the data. Is the process in statistical control?
(b) If necessary, revise the control limits computed in part (a), assuming that any samples that plot outside the control limits can be eliminated.

Day	$\bar{x}$	r	Day	$\bar{x}$	r
1	5.45	1.21	14	7.01	1.45
2	5.39	0.95	15	5.83	1.37
3	6.85	1.43	16	6.35	1.04
4	6.74	1.29	17	6.05	0.83
5	5.83	1.35	18	7.11	1.35
6	7.22	0.88	19	7.32	1.09
7	6.39	0.92	20	5.90	1.22
8	6.50	1.13	21	5.50	0.98
9	7.15	1.25	22	6.32	1.21
10	5.92	1.05	23	6.55	0.76
11	6.45	0.98	24	5.90	1.20
12	5.38	1.36	25	5.95	1.19
13	6.03	0.83			

16-6 CONTROL CHARTS FOR INDIVIDUAL MEASUREMENTS

In many situations, the sample size used for process control is $n = 1$; that is, the sample consists of an individual unit. Some examples of these situations are as follows:

1. Automated inspection and measurement technology is used, and every unit manufactured is analyzed.
2. The production rate is very slow, and it is inconvenient to allow sample sizes of $n > 1$ to accumulate before being analyzed.
3. Repeat measurements on the process differ only because of laboratory or analysis error, as in many chemical processes.

4. In process plants, such as papermaking, measurements on some parameters such as coating thickness *across* the roll will differ very little and produce a standard deviation that is much too small if the objective is to control coating thickness *along* the roll.

In such situations, the **individuals control chart** is useful. The control chart for individuals uses the **moving range** of two successive observations to estimate the process variability. The moving range is defined as $MR_i = |X_i - X_{i-1}|$.

An estimate of σ is

$$\hat{\sigma} = \frac{\overline{MR}}{d_2} = \frac{\overline{MR}}{1.128} \tag{16-18}$$

because $d_2 = 1.128$ when two consecutive observations are used to calculate a moving range. It is also possible to establish a control chart on the moving range using D_3 and D_4 for $n = 2$. The parameters for these charts are defined as follows.

Individuals Control Chart

The center line and upper and lower control limits for a control chart for individuals are

$$\begin{aligned} UCL &= \bar{x} + 3\frac{\overline{mr}}{d_2} = \bar{x} + 3\frac{\overline{mr}}{1.128} \\ CL &= \bar{x} \\ LCL &= \bar{x} - 3\frac{\overline{mr}}{d_2} = \bar{x} - 3\frac{\overline{mr}}{1.128} \end{aligned} \tag{16-19}$$

and for a control chart for moving ranges

$$\begin{aligned} UCL &= D_4\overline{mr} = 3.267\,\overline{mr} \\ CL &= \overline{mr} \\ LCL &= D_3\overline{mr} = 0 \end{aligned}$$

The procedure is illustrated in the following example.

EXAMPLE 16-2

Table 16-3 shows 20 observations on concentration for the output of a chemical process. The observations are taken at one-hour intervals. If several observations are taken at the same time, the observed concentration reading will differ only because of measurement error. Since the measurement error is small, only one observation is taken each hour.

To set up the control chart for individuals, note that the sample average of the 20 concentration readings is $\bar{x} = 99.1$ and that the average of the moving ranges of two observations shown in the last column of Table 16-3 is $\overline{mr} = 2.59$. To set up the moving-range chart, we note that $D_3 = 0$ and $D_4 = 3.267$ for $n = 2$. Therefore, the moving-range chart has center line $\overline{mr} = 2.59$, $LCL = 0$, and $UCL = D_4\overline{mr} = (3.267)(2.59) = 8.46$. The control chart is shown as the lower control chart in Fig. 16-11 on page 618. This control chart was constructed by Minitab. Because no points exceed the upper control limit, we may now set up the control chart for individual concentration measurements. If a moving range of $n = 2$ observations is used,

$d_2 = 1.128$. For the data in Table 16-3 we have

$$UCL = \bar{x} + 3\frac{\overline{mr}}{d_2} = 99.1 + 3\frac{2.59}{1.128} = 105.99$$

$$CL = \bar{x} = 99.1$$

$$LCL = \bar{x} - 3\frac{\overline{mr}}{d_2} = 99.1 - 3\frac{2.59}{1.128} = 92.21$$

The control chart for individual concentration measurements is shown as the upper control chart in Fig. 16-11. There is no indication of an out-of-control condition.

The chart for individuals can be interpreted much like an ordinary $\overline{X}$ control chart. A shift in the process average will result in either a point (or points) outside the control limits, or a pattern consisting of a run on one side of the center line.

Some care should be exercised in interpreting patterns on the moving-range chart. The moving ranges are correlated, and this correlation may often induce a pattern of runs or cycles on the chart. The individual measurements are assumed to be uncorrelated, however, and any apparent pattern on the individuals' control chart should be carefully investigated.

The control chart for individuals is very insensitive to small shifts in the process mean. For example, if the size of the shift in the mean is one standard deviation, the average number of points to detect this shift is 43.9. This result is shown later in the chapter. While the performance of the control chart for individuals is much better for large shifts, in many situations the shift of interest is not large and more rapid shift detection is desirable. In these cases, we recommend the *cumulative sum control chart* (discussed in Section 16-10) or an *exponentially weighted moving-average chart* (Montgomery, 2001).

Table 16-3 Chemical Process Concentration Measurements

Observation	Concentration x	Moving Range mr
1	102.0	
2	94.8	7.2
3	98.3	3.5
4	98.4	0.1
5	102.0	3.6
6	98.5	3.5
7	99.0	0.5
8	97.7	1.3
9	100.0	2.3
10	98.1	1.9
11	101.3	3.2
12	98.7	2.6
13	101.1	2.4
14	98.4	2.7
15	97.0	1.4
16	96.7	0.3
17	100.3	3.6
18	101.4	1.1
19	97.2	4.2
20	101.0	3.8
	$\bar{x} = 99.1$	$\overline{mr} = 2.59$

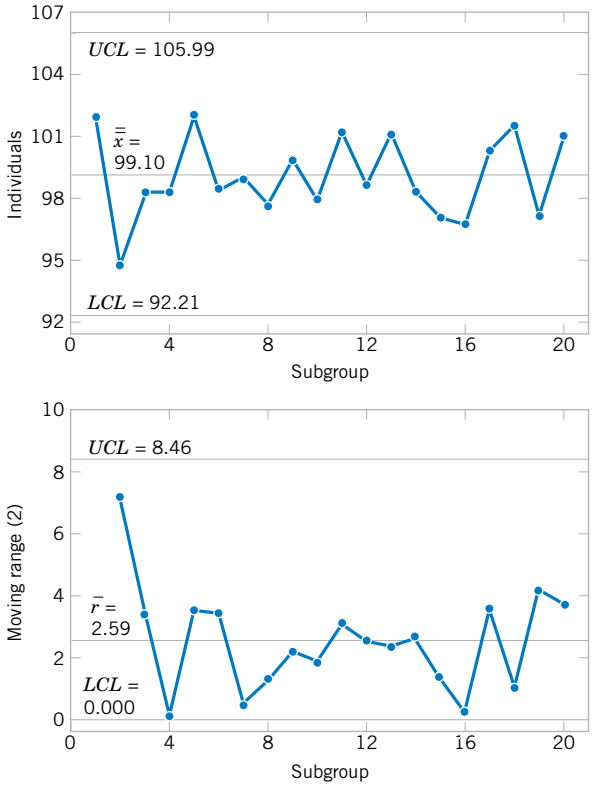

Figure 16-11 Control charts for individuals and the moving range (from Minitab) for the chemical process concentration data.

Some individuals have suggested that limits narrower than 3-sigma be used on the chart for individuals to enhance its ability to detect small process shifts. This is a dangerous suggestion, for narrower limits will dramatically increase false alarms such that the charts may be ignored and become useless. If you are interested in detecting small shifts, use the cumulative sum or exponentially weighted moving-average control chart referred to on the previous page.

EXERCISES FOR SECTION 16-6

16-9. Twenty successive hardness measurements are made on a metal alloy, and the data are shown in the following table.

Observation	Hardness	Observation	Hardness
1	51	11	51
2	52	12	57
3	54	13	58
4	55	14	50
5	55	15	53
6	51	16	52
7	52	17	54
8	50	18	50
9	51	19	56
10	56	20	53

(a) Using all the data, compute trial control limits for individual observations and moving-range charts. Construct the chart and plot the data. Determine whether the process is in statistical control. If not, assume assignable causes can be found to eliminate these samples and revise the control limits.

(b) Estimate the process mean and standard deviation for the in-control process.

16-10. In a semiconductor manufacturing process CVD metal thickness was measured on 30 wafers obtained over approximately two weeks. Data are shown in the following table.

(a) Using all the data, compute trial control limits for individual observations and moving-range charts. Construct the chart and plot the data. Determine whether the process is in statistical control. If not, assume assignable causes can be found to eliminate these samples and revise the control limits.

(b) Estimate the process mean and standard deviation for the in-control process.

Wafer	x	Wafer	x
1	16.8	16	15.4
2	14.9	17	14.3
3	18.3	18	16.1
4	16.5	19	15.8
5	17.1	20	15.9
6	17.4	21	15.2
7	15.9	22	16.7
8	14.4	23	15.2
9	15.0	24	14.7
10	15.7	25	17.9
11	17.1	26	14.8
12	15.9	27	17.0
13	16.4	28	16.2
14	15.8	29	15.6
15	15.4	30	16.3

16-11. The diameter of holes is measured in consecutive order by an automatic sensor. The results of measuring 25 holes are in the following table.

Sample	Diameter	Sample	Diameter
1	9.94	14	9.99
2	9.93	15	10.12
3	10.09	16	9.81
4	9.98	17	9.73
5	10.11	18	10.14
6	9.99	19	9.96
7	10.11	20	10.06
8	9.84	21	10.11
9	9.82	22	9.95
10	10.38	23	9.92
11	9.99	24	10.09
12	10.41	25	9.85
13	10.36		

(a) Using all the data, compute trial control limits for individual observations and moving-range charts. Construct the control chart and plot the data. Determine whether the process is in statistical control. If not, assume assignable causes can be found to eliminate these samples and revise the control limits.

(b) Estimate the process mean and standard deviation for the in-control process.

16-12. The viscosity of a chemical intermediate is measured every hour. Twenty samples each of size $n = 1$, are in the following table.

Sample	Viscosity
1	495
2	491
3	501
4	501
5	512
6	540
7	492
8	504
9	542
10	508
11	493
12	507
13	503
14	475
15	497
16	499
17	468
18	486
19	511
20	487

(a) Using all the data, compute trial control limits for individual observations and moving-range charts. Determine whether the process is in statistical control. If not, assume assignable causes can be found to eliminate these samples and revise the control limits.

(b) Estimate the process mean and standard deviation for the in-control process.

16-7 PROCESS CAPABILITY

It is usually necessary to obtain some information about the **process capability,** that is, the performance of the process when it is operating in control. Two graphical tools, the **tolerance chart** (or tier chart) and the **histogram,** are helpful in assessing process capability.

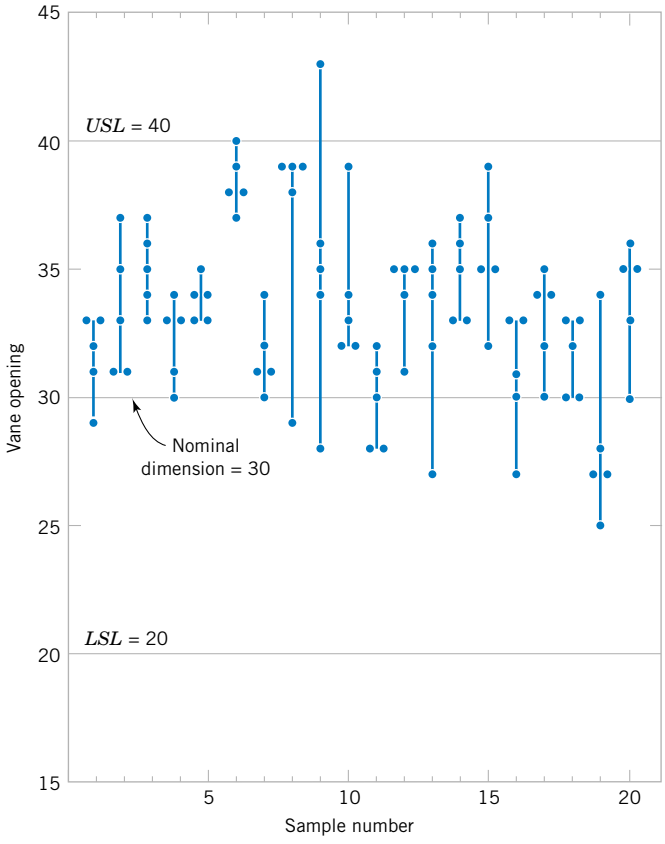

Figure 16-12
Tolerance diagram of vane openings.

The tolerance chart for all 20 samples from the vane-manufacturing process is shown in Fig. 16-12. The specifications on vane opening are 0.5030 ± 0.0010 in. In terms of the coded data, the upper specification limit is $USL = 40$ and the lower specification limit is $LSL = 20$, and these limits are shown on the chart in Fig. 16-12. Each measurement is plotted on the tolerance chart. Measurements from the same subgroup are connected with lines. The tolerance chart is useful in revealing patterns over time in the individual measurements, or it may show that a particular value of $\bar{x}$ or r was produced by one or two unusual observations in the sample. For example, note the two unusual observations in sample 9 and the single unusual observation in sample 8. Note also that it is appropriate to plot the specification limits on the tolerance chart, since it is a chart of individual measurements. **It is never appropriate to plot specification limits on a control chart or to use the specifications in determining the control limits.** Specification limits and control limits are unrelated. Finally, note from Fig. 16-12 that the process is running off-center from the nominal dimension of 30 (or 0.5030 inch).

The histogram for the vane-opening measurements is shown in Fig. 16-13. The observations from samples 6, 8, 9, 11, and 19 (corresponding to out of-control points on either the $\overline{X}$ or R chart) have been deleted from this histogram. The general impression from examining this histogram is that the process is capable of meeting the specification but that it is running off-center.

Another way to express process capability is in terms of an index that is defined as follows.

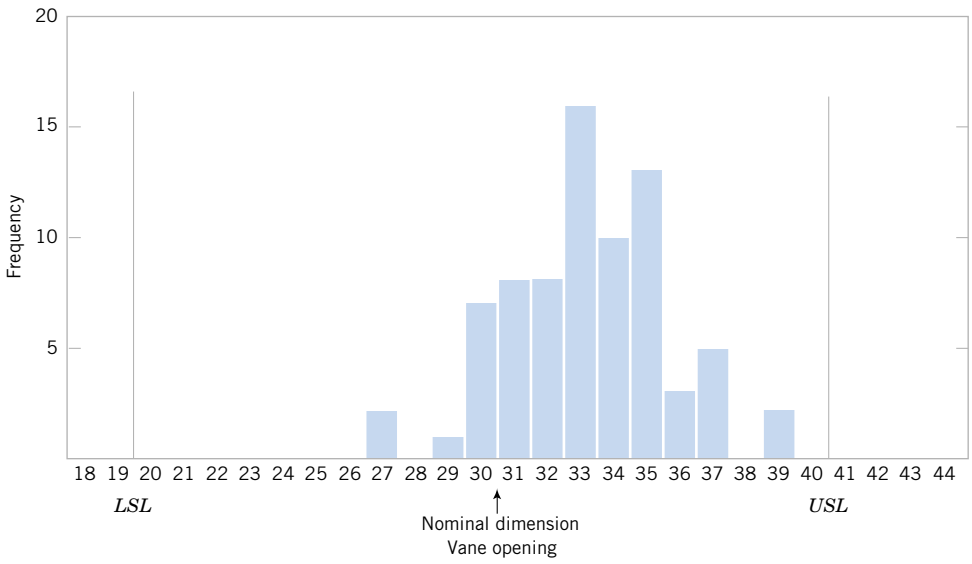

Figure 16-13
Histogram for vane opening.

Process Capability Ratio

The **process capability ratio** (*PCR*) is

$$PCR = \frac{USL - LSL}{6\sigma} \tag{16-20}$$

The numerator of *PCR* is the width of the specifications. The limits 3σ on either side of the process mean are sometimes called **natural tolerance limits,** for these represent limits that an in-control process should meet with most of the units produced. Consequently, 6σ is often referred to as the width of the process. For the vane opening, where our sample size is 5, we could estimate σ as

$$\hat{\sigma} = \frac{\bar{r}}{d_2} = \frac{5.0}{2.326} = 2.15$$

Therefore, the *PCR* is estimated to be

$$PCR = \frac{USL - LSL}{6\hat{\sigma}} = \frac{40 - 20}{6(2.15)} = 1.55$$

The *PCR* has a natural interpretation: $(1/PCR)100$ is just the percentage of the specifications' width used by the process. Thus, the vane-opening process uses approximately $(1/1.55)100 = 64.5\%$ of the specifications' width.

Figure 16-14(a) shows a process for which the *PCR* exceeds unity. Since the process natural tolerance limits lie inside the specifications, very few defective or nonconforming units will be produced. If $PCR = 1$, as shown in Fig. 16-14(b), more nonconforming units result. In fact, for a normally distributed process, if $PCR = 1$, the fraction

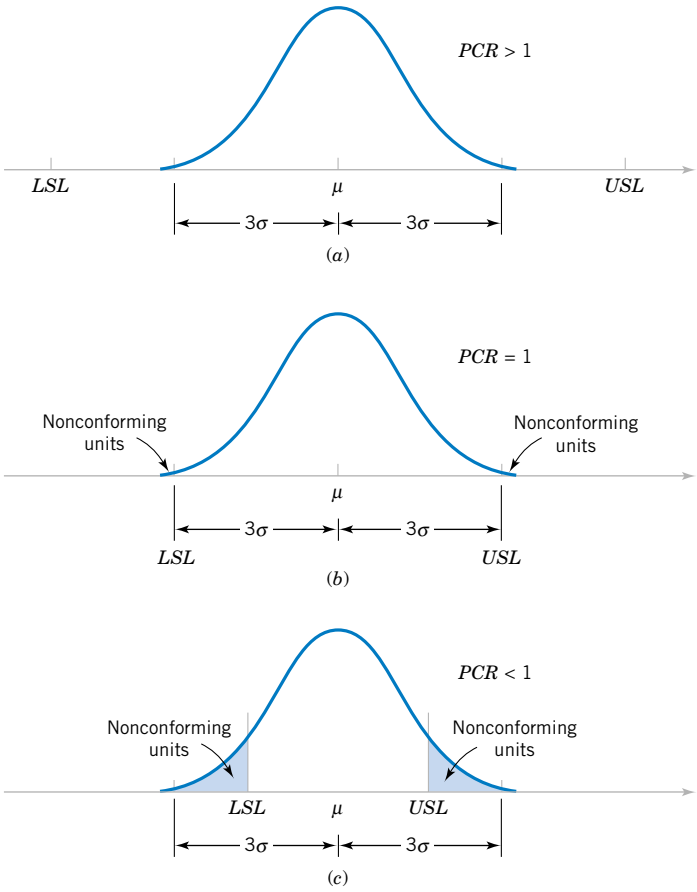

Figure 16-14
Process fallout and the process capability ratio (PCR).

nonconforming is 0.27%, or 2700 parts per million. Finally, when the *PCR* is less than unity, as in Fig. 16-14(c), the process is very yield-sensitive and a large number of nonconforming units will be produced.

The definition of the *PCR* given in Equation 16-19 implicitly assumes that the process is centered at the nominal dimension. If the process is running off-center, its **actual capability** will be less than indicated by the *PCR*. It is convenient to think of *PCR* as a measure of **potential capability,** that is, capability with a centered process. If the process is not centered, a measure of actual capability is often used. This ratio, called PCR_k, is defined below.

***PCR*$_k$**

$$PCR_k = \min\left[\frac{USL - \mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right] \qquad (16\text{-}21)$$

In effect, PCR_k is a one-sided process capability ratio that is calculated relative to the specification limit nearest to the process mean. For the vane-opening process, we find that the

estimate of the process capability ratio PCR_k is

$$\widehat{PCR}_k = \min\left[\frac{USL - \bar{\bar{x}}}{3\hat{\sigma}}, \frac{\bar{\bar{x}} - LSL}{3\hat{\sigma}}\right]$$
$$= \min\left[\frac{40 - 33.19}{3(2.15)} = 1.06,\ \frac{33.19 - 20}{3(2.15)} = 2.04\right] = 1.06$$

Note that if $PCR = PCR_k$, the process is centered at the nominal dimension. Since $\widehat{PCR}_k = 1.06$ for the vane-opening process and $\widehat{PCR} = 1.55$, the process is obviously running off-center, as was first noted in Figs. 16-14 and 16-17. This off-center operation was ultimately traced to an oversized wax tool. Changing the tooling resulted in a substantial improvement in the process (Montgomery, 2001).

The fractions of nonconforming output (or fallout) below the lower specification limit and above the upper specification limit are often of interest. Suppose that the output from a normally distributed process in statistical control is denoted as X. The fractions are determined from

$$P(X < LSL) = P(Z < (LSL - \mu)/\sigma) \qquad P(X > USL) = P(Z > (USL - \mu)/\sigma)$$

EXAMPLE 16-3

For an electronic manufacturing process a current has specifications of 100 ± 10 milliamperes. The process mean μ and standard deviation σ are 107.0 and 1.5, respectively. The process mean is nearer to the USL. Consequently,

$$PCR = (110 - 90)/(6 \cdot 1.5) = 2.22 \quad \text{and} \quad PCR_k = (110 - 107)/(3 \cdot 1.5) = 0.67$$

The small PCR_k indicates that the process is likely to produce currents outside of the specification limits. From the normal distribution in Appendix Table II

$$P(X < LSL) = P(Z < (90 - 107)/1.5) = P(Z < -11.33) = 0$$
$$P(X > USL) = P(Z > (110 - 107)/1.5) = P(Z > 2) = 0.023$$

For this example, the relatively large probability of exceeding the USL is a warning of potential problems with this criterion even if none of the measured observations in a preliminary sample exceed this limit. We emphasize that the fraction-nonconforming calculation assumes that the observations are normally distributed and the process is in control. Departures from normality can seriously affect the results. The calculation should be interpreted as an approximate guideline for process performance. To make matters worse, μ and σ need to be estimated from the data available and a small sample size can result in poor estimates that further degrade the calculation.

Montgomery (2001) provides guidelines on appropriate values of the PCR and a table relating fallout for a normally distributed process in statistical control to the value of PCR. Many U.S. companies use $PCR = 1.33$ as a minimum acceptable target and $PCR = 1.66$ as a minimum target for strength, safety, or critical characteristics. Some companies require that internal processes and those at suppliers achieve a $PCR_k = 2.0$. Figure 16-15 illustrates a process with $PCR = PCR_k = 2.0$. Assuming a normal distribution, the calculated fallout for this process is 0.0018 parts per million. A process with $PCR_k = 2.0$ is referred to as a **six-sigma process** because the distance from the process mean to the nearest specification is six standard deviations. The reason that such a large process capability is often required is that it

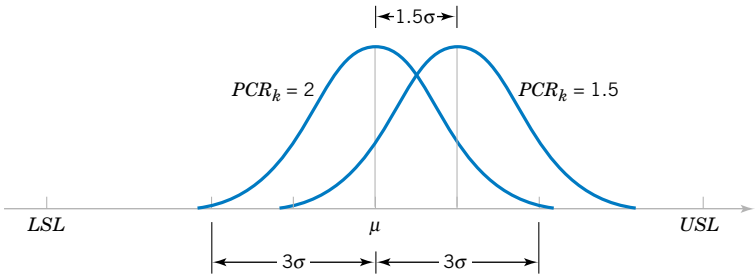

Figure 16-15 Mean of a six-sigma process shifts by 1.5 standard deviations.

is difficult to maintain a process mean at the center of the specifications for long periods of time. A common model that is used to justify the importance of a six-sigma process is illustrated by referring to Fig. 16-15. If the process mean shifts off-center by 1.5 standard deviations, the PCR_k decreases to $4.5\sigma/3\sigma = 1.5$. Assuming a normally distributed process, the fallout of the shifted process is **3.4 parts per million.** Consequently, the mean of a 6-sigma process can shift 1.5 standard deviations from the center of the specifications and still maintain a fallout of 3.4 parts per million.

In addition, some U.S. companies, particularly the automobile industry, have adopted the terminology $C_p = PCR$ and $C_{pk} = PCR_k$. Because C_p has another meaning in statistics (in multiple regression) we prefer the traditional notation PCR and PCR_k.

We repeat that process capability calculations are meaningful only for stable processes; that is, processes that are in control. A process capability ratio indicates whether or not the natural or chance variability in a process is acceptable relative to the specifications.

EXERCISES FOR SECTION 16-7

16-13. A normally distributed process uses 66.7% of the specification band. It is centered at the nominal dimension, located halfway between the upper and lower specification limits.
(a) Estimate PCR and PCR_k. Interpret these ratios.
(b) What fallout level (fraction defective) is produced?

16-14. Reconsider Exercise 16-1. Use the revised control limits and process estimates.
(a) Estimate PCR and PCR_k. Interpret these ratios.
(b) What percentage of defectives is being produced by this process?

16-15. Reconsider Exercise 16-2, where the specification limits are 14.50 ± 0.50.
(a) What conclusions can you draw about the ability of the process to operate within these limits? Estimate the percentage of defective items that will be produced.
(b) Estimage PCR and PCR_k. Interpret these ratios.

16-16. Reconsider Exercise 16-3. Using the process estimates, what is the fallout level if the coded specifications are 10 ± 5 mm? Estimate PCR and interpret this ratio.

16-17. A normally distributed process uses 85% of the specification band. It is centered at the nominal dimension, located halfway between the upper and lower specification limits.
(a) Estimate PCR and PCR_k. Interpret these ratios.
(b) What fallout level (fraction defective) is produced?

16-18. Reconsider Exercise 16-5. Suppose that the quality characteristic is normally distributed with specification at 220 ± 40. What is the fallout level? Estimate PCR and PCR_k and interpret these ratios.

16-19. Reconsider Exercise 16-6. Suppose that the variable is normally distributed with specifications at 220 ± 50. What is the proportion out of specifications? Estimate and interpret PCR and PCR_k.

16-20. Reconsider Exercise 16-4(a). Assuming that both charts exhibit statistical control and that the process specifications are at 20 ± 5, estimate PCR and PCR_k and interpret these ratios.

16-21. Reconsider Exercise 16-8. Given that the specifications are at 6.0 ± 1.0, estimate PCR and PCR_k and interpret these ratios.

16-22. Reconsider 16-7(b). What are the natural tolerance limits of this process?

16-23. Reconsider 16-12. The viscosity specifications are at 500 ± 25. Calculate estimates of the process capability ratios PCR and PCR_k for this process and provide an interpretation.

16-8 ATTRIBUTE CONTROL CHARTS

16-8.1 *P* Chart (Control Chart for Proportions)

Often it is desirable to classify a product as either defective or nondefective on the basis of comparison with a standard. This classification is usually done to achieve economy and simplicity in the inspection operation. For example, the diameter of a ball bearing may be checked by determining whether it will pass through a gauge consisting of circular holes cut in a template. This kind of measurement would be much simpler than directly measuring the diameter with a device such as a micrometer. Control charts for attributes are used in these situations. Attribute control charts often require a considerably larger sample size than do their variable measurements counterparts. In this section, we will discuss the **fraction-defective control chart,** or ***P* chart.** Sometimes the *P* chart is called the **control chart for fraction nonconforming.**

Suppose D is the number of defective units in a random sample of size n. We assume that D is a binomial random variable with unknown parameter p. The fraction defective

$$\hat{P} = \frac{D}{n}$$

of each sample is plotted on the chart. Furthermore, the variance of the statistic $\hat{P}$ is

$$\sigma^2_{\hat{P}} = \frac{p(1-p)}{n}$$

Therefore, a P chart for fraction defective could be constructed using p as the center line and control limits at

$$UCL = p + 3\sqrt{\frac{p(1-p)}{n}} \qquad LCL = p - 3\sqrt{\frac{p(1-p)}{n}} \tag{16-22}$$

However, the true process fraction defective is almost always unknown and must be estimated using the data from preliminary samples.

Suppose that m preliminary samples each of size n are available, and let D_i be the number of defectives in the ith sample. The $\hat{P}_i = D_i/n$ is the sample fraction defective in the ith sample. The average fraction defective is

$$\bar{P} = \frac{1}{m}\sum_{i=1}^{m}\hat{P}_i = \frac{1}{mn}\sum_{i=1}^{m}D_i \tag{16-23}$$

Now $\bar{P}$ may be used as an estimator of p in the center line and control limit calculations.

***P* Chart**

The center line and upper and lower control limits for the P chart are

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \quad CL = \bar{p} \quad LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \tag{16-24}$$

where $\bar{p}$ is the observed value of the average fraction defective.

Table 16-4 Number of Defectives in Samples of 100 Ceramic Substrates

Sample	No. of Defectives	Sample	No. of Defectives
1	44	11	36
2	48	12	52
3	32	13	35
4	50	14	41
5	29	15	42
6	31	16	30
7	46	17	46
8	52	18	38
9	44	19	26
10	48	20	30

These control limits are based on the normal approximation to the binomial distribution. When p is small, the normal approximation may not always be adequate. In such cases, we may use control limits obtained directly from a table of binomial probabilities. If $\bar{p}$ is small, the lower control limit obtained from the normal approximation may be a negative number. If this should occur, it is customary to consider zero as the lower control limit.

EXAMPLE 16-4 Suppose we wish to construct a fraction-defective control chart for a ceramic substrate production line. We have 20 preliminary samples, each of size 100; the number of defectives in each sample is shown in Table 16-4. Assume that the samples are numbered in the sequence of production. Note that $\bar{p} = (800/2000) = 0.40$; therefore, the trial parameters for the control chart are

$$UCL = 0.40 + 3\sqrt{\frac{(0.40)(0.60)}{100}} = 0.55 \qquad CL = 0.40$$

$$LCL = 0.40 - 3\sqrt{\frac{(0.40)(0.60)}{100}} = 0.25$$

The control chart is shown in Fig. 16-16. All samples are in control. If they were not, we would search for assignable causes of variation and revise the limits accordingly. This chart can be used for controlling future production.

Although this process exhibits statistical control, its defective rate ($\bar{p} = 0.40$) is very poor. We should take appropriate steps to investigate the process to determine why such a large number of defective units is being produced. Defective units should be analyzed to determine the specific types of defects present. Once the defect types are known, process changes should be investigated to determine their impact on defect levels. Designed experiments may be useful in this regard.

Computer software also produces an ***NP* chart.** This is just a control chart of $n\hat{P} = D$, the number of defectives in a sample. The points, center line, and control limits for this chart are just multiples (times n) of the corresponding elements of a P chart. The use of an NP chart avoids the fractions in a P chart.

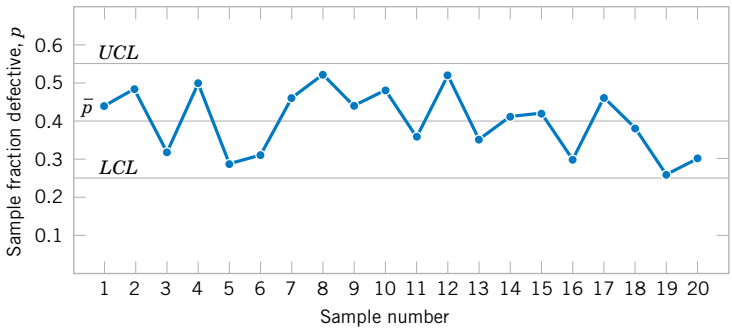

Figure 16-16 *P* chart for a ceramic substrate.

16-8.2 *U* Chart (Control Chart for Defects per Unit)

It is sometimes necessary to monitor the number of defects in a unit of product rather than the fraction defective. Suppose that in the production of cloth it is necessary to control the number of defects per yard or that in assembling an aircraft wing the number of missing rivets must be controlled. In these situations we may use the control chart for defects per unit, or the ***U* chart.** Many defects-per-unit situations can be modeled by the Poisson distribution.

If each sample consists of n units and there are C total defects in the sample,

$$U = \frac{C}{n}$$

is the average number of defects per unit. A U chart may be constructed for such data.

If the number of defects in a unit is a Poisson random variable with parameter λ, the mean and variance of this distribution are both λ. Each point on the chart is U, the average number of defects per unit from a sample of n units. Therefore, the mean of U is λ and the variance of U is λ/n.

$$UCL = \lambda + 3\sqrt{\frac{\lambda}{n}}$$

$$LCL = \lambda - 3\sqrt{\frac{\lambda}{n}} \tag{16-25}$$

If there are m preliminary samples, and the number of defects per unit in these samples are U_1, $U_2, \ldots, U_m$, the estimator of the average number of defects per unit is

$$\bar{U} = \frac{1}{m}\sum_{i=1}^{m} U_i \tag{16-26}$$

The parameters of the U chart are defined as follows.

***U* Chart**

The center line and upper and lower control limits on the U chart are

$$UCL = \bar{u} + 3\sqrt{\frac{\bar{u}}{n}} \qquad CL = \bar{u} \qquad LCL = \bar{u} - 3\sqrt{\frac{\bar{u}}{n}} \tag{16-27}$$

where $\bar{u}$ is the average number of defects per unit.

These control limits are based on the normal approximation to the Poisson distribution. When λ is small, the normal approximation may not always be adequate. In such cases, we may use control limits obtained directly from a table of Poisson probabilities. If $\bar{u}$ is small, the lower control limit obtained from the normal approximation may be a negative number. If this should occur, it is customary to consider zero as the lower control limit.

EXAMPLE 16-5

Printed circuit boards are assembled by a combination of manual assembly and automation. A flow solder machine is used to make the mechanical and electrical connections of the leaded components to the board. The boards are run through the flow solder process almost continuously, and every hour five boards are selected and inspected for process-control purposes. The number of defects in each sample of five boards is noted. Results for 20 samples are shown in Table 16-5.

The center line for the *U* chart is

$$\bar{u} = \frac{1}{20}\sum_{i=1}^{20} u_i = \frac{32}{20} = 1.6$$

and the upper and lower control limits are

$$UCL = \bar{u} + 3\sqrt{\frac{\bar{u}}{n}} = 1.6 + 3\sqrt{\frac{1.6}{5}} = 3.3$$

$$LCL = \bar{u} - 3\sqrt{\frac{\bar{u}}{n}} = 1.6 - 3\sqrt{\frac{1.6}{5}} < 0$$

The control chart is plotted in Fig. 16-17. Because *LCL* is negative, it is set to 0. From the control chart in Fig. 16-17, we see that the process is in control. However, eight defects per group of five circuit boards are too many (about $8/5 = 1.6$ defects/board), and the process needs improvement. An investigation needs to be made of the specific types of defects found on the printed circuit boards. This will usually suggest potential avenues for process improvement.

Computer software also produces a ***C* chart.** This is just a control chart of *C*, the total of defects in a sample. The points, center line, and control limits for this chart are just multiples

Table 16-5 Number of Defects in Samples of Five Printed Circuit Boards

Sample	Number of Defects	Defects per Unit u_i	Sample	Number of Defects	Defects per Unit u_i
1	6	1.2	11	9	1.8
2	4	0.8	12	15	3.0
3	8	1.6	13	8	1.6
4	10	2.0	14	10	2.0
5	9	1.8	15	8	1.6
6	12	2.4	16	2	0.4
7	16	3.2	17	7	1.4
8	2	0.4	18	1	0.2
9	3	0.6	19	7	1.4
10	10	2.0	20	13	2.6

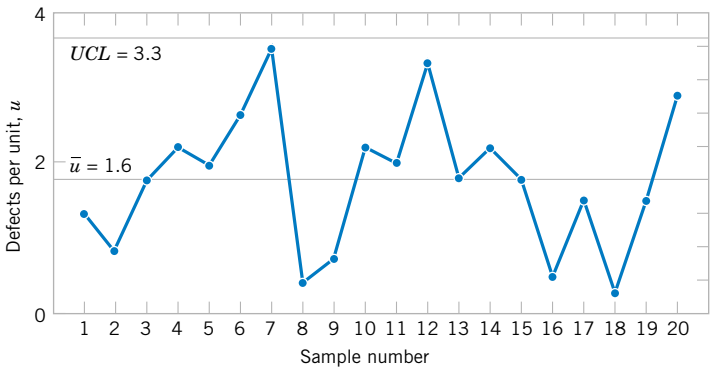

Figure 16-17 *U* chart of defects per unit on printed circuit boards.

(times n) of the corresponding elements of a U chart. The use of a C chart avoids the fractions that can occur in a U chart.

EXERCISES FOR SECTION 16-8

16-24. Suppose the following fraction defective has been found in successive samples of size 100 (read down):

0.09	0.03	0.12
0.10	0.05	0.14
0.13	0.13	0.06
0.08	0.10	0.05
0.14	0.14	0.14
0.09	0.07	0.11
0.10	0.06	0.09
0.15	0.09	0.13
0.13	0.08	0.12
0.06	0.11	0.09

(a) Using all the data, compute trial control limits for a fraction-defective control chart, construct the chart, and plot the data.

(b) Determine whether the process is in statistical control. If not, assume assignable causes can be found and out-of-control points eliminated. Revise the control limits.

16-25. The following represent the number of solder defects observed on 24 samples of five printed circuit boards: 7, 6, 8, 10, 24, 6, 5, 4, 8, 11, 15, 8, 4, 16, 11, 12, 8, 6, 5, 9, 7, 14, 8, 21.

(a) Using all the data, compute trial control limits for a U control chart, construct the chart, and plot the data.

(b) Can we conclude that the process is in control using a U chart? If not, assume assignable causes can be found, list points and revise the control limits.

16-26. The following represent the number of defects per 1000 feet in rubber-covered wire: 1, 1, 3, 7, 8, 10, 5, 13, 0, 19, 24, 6, 9, 11, 15, 8, 3, 6, 7, 4, 9, 20, 11, 7, 18, 10, 6, 4, 0, 9, 7, 3, 1, 8, 12. Do the data come from a controlled process?

16-27. Consider the data in Exercise 16-25. Set up a C chart for this process. Compare it to the U chart in Exercise 16-25. Comment on your findings.

16-28. The following are the numbers of defective solder joints found during successive samples of 500 solder joints:

Day	No. of Defectives	Day	No. of Defectives
1	106	12	37
2	116	13	25
3	164	14	88
4	89	15	101
5	99	16	64
6	40	17	51
7	112	18	74
8	36	19	71
9	69	20	43
10	74	21	80
11	42		

(a) Using all the data, compute trial control limits for a fraction-defective control chart, construct the chart, and plot the data.

(b) Determine whether the process is in statistical control. If not, assume assignable causes can be found and out-of-control points eliminated. Revise the control limits.

16-9 CONTROL CHART PERFORMANCE

Specifying the control limits is one of the critical decisions that must be made in designing a control chart. By moving the control limits further from the center line, we decrease the risk of a type I error—that is, the risk of a point falling beyond the control limits, indicating an out-of-control condition when no assignable cause is present. However, widening the control limits will also increase the risk of a type II error—that is, the risk of a point falling between the control limits when the process is really out of control. If we move the control limits closer to the center line, the opposite effect is obtained: The risk of type I error is increased, while the risk of type II error is decreased.

The control limits on a Shewhart control chart are customarily located a distance of plus or minus three standard deviations of the variable plotted on the chart from the center line. That is, the constant k in equation 16-1 should be set equal to 3. These limits are called **3-sigma control limits.**

A way to evaluate decisions regarding sample size and sampling frequency is through the **average run length (ARL)** of the control chart. Essentially, the ARL is the average number of points that must be plotted before a point indicates an out-of-control condition. For any Shewhart control chart, the ARL can be calculated from the mean of a geometric random variable (Montgomery 2001). Suppose that p is the probability that any point exceeds the control limits. Then

$$\text{ARL} = \frac{1}{p} \tag{16-28}$$

Thus, for an $\overline{X}$ chart with 3-sigma limits, $p = 0.0027$ is the probability that a single point falls outside the limits when the process is in control, so

$$\text{ARL} = \frac{1}{p} = \frac{1}{0.0027} \cong 370$$

is the average run length of the $\overline{X}$ chart when the process is in control. That is, even if the process remains in control, an out-of-control signal will be generated every 370 points, on the average.

Consider the piston ring process discussed in Section 16-4.2, and suppose we are sampling every hour. Thus, we will have a **false alarm** about every 370 hours on the average. Suppose we are using a sample size of $n = 5$ and that when the process goes out of control the mean shifts to 74.0135 millimeters. Then, the probability that $\overline{X}$ falls between the control limits of Fig. 16-3 is equal to

$$\begin{aligned} &P[73.9865 \le \overline{X} \le 74.0135 \text{ when } \mu = 74.0135] \\ &\quad = P\left[\frac{73.9865 - 74.0135}{0.0045} \le Z \le \frac{74.0135 - 74.0135}{0.0045}\right] \\ &\quad = P[-6 \le Z \le 0] = 0.5 \end{aligned}$$

Therefore, p in Equation 16-28 is 0.50, and the out-of-control ARL is

$$\text{ARL} = \frac{1}{p} = \frac{1}{0.5} = 2$$

Table 16-6 Average Run Length (ARL) for an $\overline{X}$ Chart with 3-Sigma Control Limits

Magnitude of Process Shift	ARL $n = 1$	ARL $n = 4$
0	370.4	370.4
0.5σ	155.2	43.9
1.0σ	43.9	6.3
1.5σ	15.0	2.0
2.0σ	6.3	1.2
3.0σ	2.0	1.0

That is, the control chart will require two samples to detect the process shift, on the average, so two hours will elapse between the shift and its detection (*again on the average*). Suppose this approach is unacceptable, because production of piston rings with a mean diameter of 74.0135 millimeters results in excessive scrap costs and delays final engine assembly. How can we reduce the time needed to detect the out-of-control condition? One method is to sample more frequently. For example, if we sample every half hour, only one hour will elapse (on the average) between the shift and its detection. The second possibility is to increase the sample size. For example, if we use $n = 10$, the control limits in Fig. 16-3 narrow to 73.9905 and 74.0095. The probability of $\overline{X}$ falling between the control limits when the process mean is 74.0135 millimeters is approximately 0.1, so $p = 0.9$, and the out-of-control ARL is

$$\text{ARL} = \frac{1}{p} = \frac{1}{0.9} = 1.11$$

Thus, the larger sample size would allow the shift to be detected about twice as quickly as the old one. If it became important to detect the shift in the first hour after it occurred, two control chart designs would work:

Design 1	Design 2
Sample size: $n = 5$	Sample size: $n = 10$
Sampling frequency: every half hour	Sampling frequency: every hour

Table 16-6 provides average run lengths for an $\overline{X}$ chart with 3-sigma control limits. The average run lengths are calculated for shifts in the process mean from 0 to 3.0σ and for sample sizes of $n = 1$ and $n = 4$ by using $1/p$, where p is the probability that a point plots outside of the control limits. Figure 16-18 illustrates a shift in the process mean of 2σ.

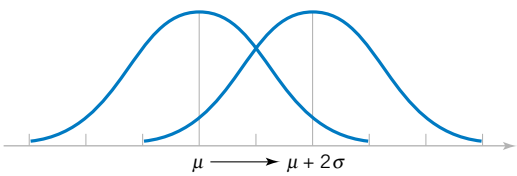

Figure 16-18 Process mean shift of 2σ.

EXERCISES FOR SECTION 16-9

16-29. Consider the $\overline{X}$ control chart in Fig. 16-3. Suppose that the mean shifts to 74.010 millimeters.
(a) What is the probability that this shift will be detected on the next sample?
(b) What is the ARL after the shift?

16-30. An $\overline{X}$ chart uses samples of size 4. The center line is at 100, and the upper and lower 3-sigma control limits are at 106 and 94, respectively.
(a) What is the process σ?
(b) Suppose the process mean shifts to 96. Find the probability that this shift will be detected on the next sample.
(c) Find the ARL to detect the shift in part (b).

16-31. Consider the revised $\overline{X}$ control chart in Exercise 16-1 with $\hat{\sigma} = 2.466$, $UCL = 37.404$, $LCL = 30.780$, and $n = 5$. Suppose that the mean shifts to 36.
(a) What is the probability that this shift will be detected on the next sample?
(b) What is the ARL after the shift?

16-32. Consider the $\overline{X}$ control chart in Exercise 16-2(a) with $\bar{r} = 0.344$, $UCL = 14.708$, $LCL = 14.312$, and $n = 5$. Suppose that the mean shifts to 14.6.
(a) What is the probability that this shift will be detected on the next sample?
(b) What is the ARL after the shift?

16-33. Consider the $\overline{X}$ control chart in Exercise 16-3(a) with $\bar{r} = 6.750$, $UCL = 15.630$, $LCL = 5.795$, and $n = 4$. Suppose that the mean shifts to 13.
(a) What is the probability that this shift will be detected on the next sample?
(b) What is the ARL after the shift?

16-34. Consider the $\overline{X}$ control chart in Exercise 16-4(a) with $\hat{\sigma} = 1.40$, $UCL = 21.88$, $LCL = 18.12$, and $n = 5$. Suppose that the mean shifts to 17.
(a) What is the probability that this shift will be detected on the next sample?
(b) What is the ARL after the shift?

16-35. Consider the $\overline{X}$ control chart in Exercise 16-5 with $\bar{r} = 34.286$, $UCL = 242.780$, $LCL = 203.220$, and $n = 5$. Suppose that the mean shifts to 210.
(a) What is the probability that this shift will be detected on the next sample?
(b) What is the ARL after the shift?

16-36. Consider the revised $\overline{X}$ control chart in Exercise 16-7 with $\hat{\sigma} = 0.000924$, $UCL = 0.0635$, $LCL = 0.0624$, and $n = 5$. Suppose that the mean shifts to 0.0625.
(a) What is the probability that this shift will be detected on the next sample?
(b) What is the ARL after the shift?

16-37. Consider the revised $\overline{X}$ control chart in Exercise 16-8 with $\hat{\sigma} = 0.669$, $UCL = 7.443$, $LCL = 5.125$, and $n = 3$. Suppose that the mean shifts to 5.5.
(a) What is the probability that this shift will be detected on the next sample?
(b) What is the ARL after the shift?

16-10 CUMULATIVE SUM CONTROL CHART

In Sections 16-5 and 16-6 we have presented basic types of **Shewhart control charts.** A major disadvantage of any Shewhart control chart is that the chart is relatively insensitive to small shifts in the process, say, on the order of about 1.5σ or less. One reason for this relatively poor performance in detecting small process shifts is that the Shewhart chart makes use of only the information in the last plotted point, and it ignores the information in the sequence of points. This problem can be addressed, to some extent by adding criteria such as the **Western Electric rules** to a Shewhart chart, but the use of these rules reduces the simplicity and ease of interpretation of the chart. These rules would also cause the in-control average run length of a Shewhart chart to drop below 370. This increase in the false alarm rate can have serious practical consequences.

A very effective alternative to the Shewhart control chart is the **cumulative sum control chart** (or **CUSUM**). This chart has much better performance (in terms of ARL) for detecting small shifts than the Shewhart chart, but it does not cause the in-control ARL to drop significantly. This section will illustrate the use of the CUSUM for sample averages and individual measurements.

The CUSUM chart plots the cumulative sums of the deviations of the sample values from a target value. For example, suppose that samples of size $n \geq 1$ are collected, and $\overline{X}_j$ is the

average of the *j*th sample. Then if μ_0 is the target for the process mean, the cumulative sum control chart is formed by plotting the quantity

$$S_i = \sum_{j=1}^{i} (\bar{X}_j - \mu_0) \tag{16-29}$$

against the sample number i. Now, S_i is called the cumulative sum up to and including the *i*th sample. Because they combine information from *several* samples, cumulative sum charts are more effective than Shewhart charts for detecting small process shifts. Furthermore, they are particularly effective with samples of $n = 1$. This makes the cumulative sum control chart a good candidate for use in the chemical and process industries where rational subgroups are frequently of size 1, as well as in discrete parts manufacturing with automatic measurement of each part and online control using a microcomputer directly at the work center.

If the process remains in control at the target value μ_0, the cumulative sum defined in equation 16-29 should fluctuate around zero. However, if the mean shifts upward to some value $\mu_1 > \mu_0$, say, an upward or positive drift will develop in the cumulative sum S_i. Conversely, if the mean shifts downward to some $\mu_1 < \mu_0$, a downward or negative drift in S_i will develop. Therefore, if a trend develops in the plotted points either upward or downward, we should consider this as evidence that the process mean has shifted, and a search for the assignable cause should be performed.

This theory can easily be demonstrated by applying the CUSUM to the chemical process concentration data in Table 16-3. Since the concentration readings are individual measurements, we would take $\bar{X}_j = X_j$ in computing the CUSUM. Suppose that the target value for the concentration is $\mu_0 = 99$. Then the CUSUM is

$$\begin{aligned} S_i &= \sum_{j=1}^{i} (X_j - 99) \\ &= (X_i - 99) + \sum_{j=1}^{i-1} (X_j - 99) \\ &= (X_i - 99) + S_{i-1} \end{aligned}$$

Table 16-7 shows the computation of this CUSUM, where the starting value of the CUSUM, S_0, is taken to be zero. Figure 16-19 plots the CUSUM from the last column of Table 16-7. Notice that the CUSUM fluctuates around the value of 0.

The graph in Fig. 16-19 is not a control chart because it lacks control limits. There are two general approaches to devising control limits for CUSUMS. The older of these two methods is the V-mask procedure. A typical V mask is shown in Fig. 16-20(a). It is a V-shaped notch in a plane that can be placed at different locations on the CUSUM chart. The decision procedure consists of placing the V mask on the cumulative sum control chart with the point O on the last value of s_i and the line OP parallel to the horizontal axis. If all the previous cumulative sums, $s_1, s_2, \ldots, s_{i-1}$, lie within the two arms of the V mask, the process is in control. However, if any s_i lies outside the arms of the mask, the process is considered to be out of control. In actual use, the V mask would be applied to each new point on the CUSUM chart as soon as it was plotted. In the example shown in Fig. 16-20(b), an upward shift in the mean is indicated, since at least one of the points that have occurred earlier than sample 22 now lies below the lower arm of the mask, when the V mask is centered on the thirtieth observation. If the point lies above the upper arm, a downward shift in the mean is

Table 16-7 CUSUM Computations for the Chemical Process Concentration Data in Table 16-3

Observation, i	x_i	$x_i - 99$	$s_i = (x_i - 99) + s_{i-1}$
1	102.0	3.0	3.0
2	94.8	−4.2	−1.2
3	98.3	−0.7	−1.9
4	98.4	−0.6	−2.5
5	102.0	3.0	0.5
6	98.5	−0.5	0.0
7	99.0	0.0	0.0
8	97.7	−1.3	−1.3
9	100.0	1.0	−0.3
10	98.1	−0.9	−1.2
11	101.3	2.3	1.1
12	98.7	−0.3	0.8
13	101.1	2.1	2.9
14	98.4	−0.6	2.3
15	97.0	−2.0	0.3
16	96.7	−2.3	−2.0
17	100.3	1.3	−0.7
18	101.4	2.4	1.7
19	97.2	−1.8	−0.1
20	101.0	2.0	1.9

indicated. Thus, the V mask forms a visual frame of reference similar to the control limits on an ordinary Shewhart control chart. For the technical details of designing the V mask, see Montgomery (2001).

While some computer programs plot CUSUMS with the V-mask control scheme, we feel that the other approach to CUSUM control, the **tabular CUSUM,** is superior.

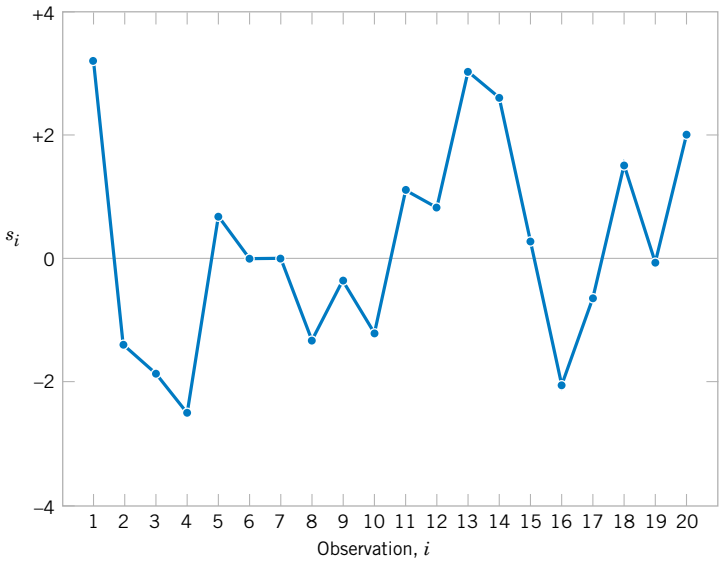

Figure 16-19 Plot of the cumulative sum for the concentration data, Table 16-7.

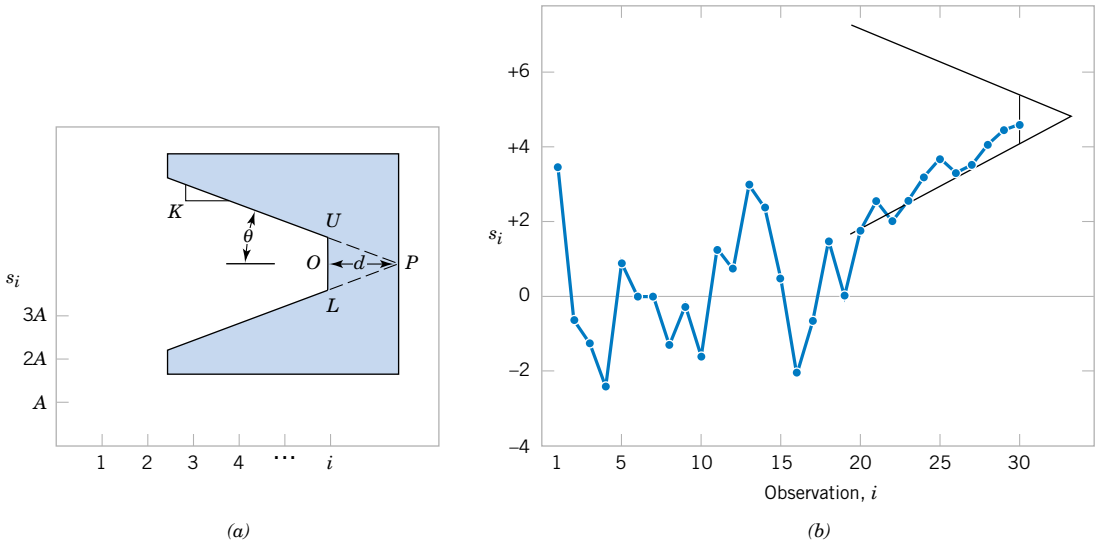

Figure 16-20 The cumulative sum control chart. (a) The V-mask and scaling. (b) The cumulative sum control chart in operation.

The tabular procedure is particularly attractive when the CUSUM is implemented on a computer.

Let $S_H(i)$ be an upper one-sided CUSUM for period i and $S_L(i)$ be a lower one-sided CUSUM for period i. These quantities are calculated from

CUSUM Control Chart

$$s_H(i) = \max[0, \bar{x}_i - (\mu_0 + K) + s_H(i-1)] \tag{16-30}$$

and

$$s_L(i) = \max[0, (\mu_0 - K) - \bar{x}_i + s_L(i-1)] \tag{16-31}$$

where the starting values $s_H(0) = s_L(0) = 0$.

In Equations 16-30 and 16-31 K is called the **reference value,** which is usually chosen about halfway between the target μ_0 and the value of the mean corresponding to the out-of-control state, $\mu_1 = \mu_0 + \Delta$. That is, K is about one-half the magnitude of the shift we are interested in, or

$$K = \frac{\Delta}{2}$$

Notice that $S_H(i)$ and $S_L(i)$ accumulate deviations from the target value that are greater than K, with both quantities reset to zero upon becoming negative. If either $S_H(i)$ or $S_L(i)$ exceeds a constant H, the process is out of control. This constant H is usually called the **decision interval.**

EXAMPLE 16-6 **A Tabular CUSUM**

We will illustrate the tabular CUSUM by applying it to the chemical process concentration data in Table 16-7. The process target is $\mu_0 = 99$, and we will use $K = 1$ as the reference value and $H = 10$ as the decision interval. The reasons for these choices will be explained later.

Table 16-8 shows the tabular CUSUM scheme for the chemical process concentration data. To illustrate the calculations, note that

$$\begin{aligned} s_H(i) &= \max[0, x_i - (\mu_0 + K) + s_H(i-1)] = \max[0, x_i - (99 + 1) + s_H(i-1)] \\ &= \max[0, x_i - 100 + s_H(i-1)] \\ s_L(i) &= \max[0, (\mu_0 - K) - x_i + s_L(i-1)] = \max[0, (99 - 1) - x_i + s_L(i-1)] \\ &= \max[0, 98 - x_i + s_L(i-1)] \end{aligned}$$

Therefore, for observation 1 the CUSUMS are

$$s_H(1) = \max[0, x_1 - 100 + s_H(0)] = \max[0, 102.0 - 100 + 0] = 2.0$$

and

$$s_L(1) = \max[0, 98 - x_1 + s_L(0)] = \max[0, 98 - 102.0 + 0] = 0$$

as shown in Table 16-8. The quantities n_H and n_L in Table 16-8 indicate the number of periods that the CUSUM $s_H(i)$ or $s_L(i)$ have been nonzero. Notice that the CUSUMS in this example never exceed the decision interval $H = 10$. We would therefore conclude that the process is in control.

When the tabular CUSUM indicates that the process is out of control, we should search for the assignable cause, take any corrective actions indicated, and restart the CUSUMS at

Table 16-8 The Tabular CUSUM for the Chemical Process Concentration Data

Observation	Upper CUSUM				Lower CUSUM		
i	x_i	$x_i - 100$	$s_H(i)$	n_H	$98 - x_i$	$s_L(i)$	n_L
1	102.0	2.0	2.0	1	−4.0	0.0	0
2	94.8	−5.2	0.0	0	3.2	3.2	1
3	98.3	−1.7	0.0	0	−0.3	2.9	2
4	98.4	−1.6	0.0	0	−0.4	2.5	3
5	102.0	2.0	2.0	1	−4.0	0.0	0
6	98.5	−1.5	0.5	2	−0.5	0.0	0
7	99.0	−1.0	0.0	0	−1.0	0.0	0
8	97.7	−2.3	0.0	0	0.3	0.3	1
9	100.0	0.0	0.0	0	−2.0	0.0	0
10	98.1	−1.9	0.0	0	−0.1	0.0	0
11	101.3	1.3	1.3	1	−3.3	0.0	0
12	98.7	−1.3	0.0	0	−0.7	0.0	0
13	101.1	1.1	1.1	1	−3.1	0.0	0
14	98.4	−1.6	0.0	0	−0.4	0.0	0
15	97.0	−3.0	0.0	0	1.0	1.0	1
16	96.7	−3.3	0.0	0	1.3	2.3	2
17	100.3	0.3	0.3	1	−2.3	0.0	0
18	101.4	1.4	1.7	2	−3.4	0.0	0
19	97.2	−2.8	0.0	0	0.8	0.8	1
20	101.0	1.0	1.0	0	−3.0	0.0	0

zero. It may be helpful to have an estimate of the new process mean following the shift. This can be computed from

$$\hat{\mu} = \begin{cases} \mu_0 + K + \dfrac{s_H(i)}{n_H}, & \text{if } s_H(i) > H \\ \mu_0 - K - \dfrac{s_L(i)}{n_L}, & \text{if } s_L(i) > H \end{cases} \tag{16-32}$$

It is also useful to present a graphical display of the tabular CUSUMS, which are sometimes called CUSUM status charts. They are constructed by plotting $s_H(i)$ and $s_L(i)$ versus the sample number. Figure 16-21 shows the CUSUM status chart for the data in Example 16-6. Each vertical bar represents the value of $s_H(i)$ and $s_L(i)$ in period i. With the decision interval plotted on the chart, the CUSUM status chart resembles a Shewhart control chart. We have also plotted the sample statistics x_i for each period on the CUSUM status chart as the solid dots. This frequently helps the user of the control chart to visualize the actual process performance that has led to a particular value of the CUSUM.

The tabular CUSUM is designed by choosing values for the reference value K and the decision interval H. We recommend that these parameters be selected to provide good average run-length values. There have been many analytical studies of CUSUM ARL performance. Based on these studies, we may give some general recommendations for selecting H and K. Define $H = h\sigma_{\bar{X}}$ and $K = k\sigma_{\bar{X}}$, where $\sigma_{\bar{X}}$ is the standard deviation of the sample variable used in forming the CUSUM (if $n = 1$, $\sigma_{\bar{X}} = \sigma_X$). Using $h = 4$ or $h = 5$ and $k = 1/2$ will generally provide a CUSUM that has good ARL properties against a shift of about $1\sigma_{\bar{X}}$ (or $1\sigma_X$) in the process mean. If much larger or smaller shifts are of interest, set $k = \delta/2$, where δ is the size of the shift in standard deviation units. Some practitioners prefer to use a standardized variable $y_i = (\bar{x}_i - \mu_0)/\sigma_{\bar{X}}$ as the basis of the CUSUM. In that case, Equations 16-30 and 16-31 become

$$s_H(i) = \max[0, y_i - K + s_H(i-1)] \qquad \text{and} \qquad s_L(i) = \max[0, K - y_i + s_L(i-1)]$$

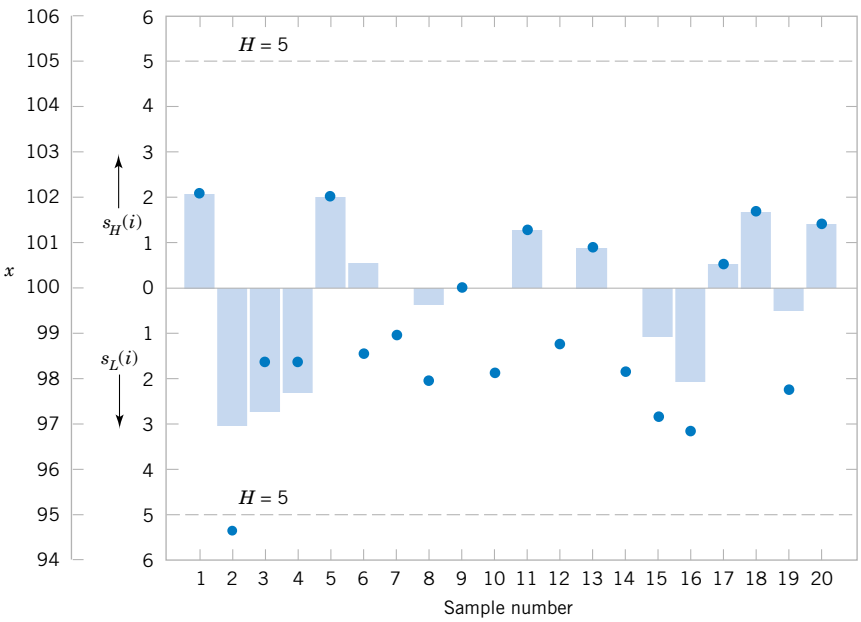

Figure 16-21 The CUSUM status chart for Example 15-6.

Table 16-9 Average Run Lengths for a CUSUM Control Chart With $K = 1/2$

Shift in Mean (multiple of $\sigma_{\bar{X}}$)	$h = 4$	$h = 5$
0	168	465
0.25	74.2	139
0.50	26.6	38.0
0.75	13.3	17.0
1.00	8.38	10.4
1.50	4.75	5.75
2.00	3.34	4.01
2.50	2.62	3.11
3.00	2.19	2.57
4.00	1.71	2.01

For this scheme, we would usually select $K = 1/2$ and $H = 4$ or $H = 5$.

To illustrate how well the recommendations of $h = 4$ or $h = 5$ with $k = 1/2$ work, consider these average run lengths in Table 16-9. Notice that a shift of $1\sigma_{\bar{X}}$ would be detected in either 8.38 samples (with $k = 1/2$ and $h = 4$) or 10.4 samples (with $k = 1/2$ and $h = 5$). By comparison, Table 16-1 shows that an $\bar{X}$ chart would require approximately 43.9 samples, on the average, to detect this shift.

These design rules were used for the CUSUM in Example 16-6. We assumed that the process standard deviation $\sigma = 2$. (This is a reasonable value; see Example 16-2.) Then with $k = 1/2$ and $h = 5$, we would use

$$K = k\sigma = \tfrac{1}{2}(2) = 1 \qquad \text{and} \qquad H = h\sigma = 5(2) = 10$$

in the tabular CUSUM procedure.

Finally, we should note that supplemental procedures such as the Western Electric rules cannot be safely applied to the CUSUM, because successive values of $S_H(i)$ and $S_L(i)$ are not independent. In fact, the CUSUM can be thought of as a weighted average, where the weights are stochastic or random. In effect, all the CUSUM values are highly correlated, thereby causing the Western Electric rules to give too many false alarms.

EXERCISES FOR SECTION 16-10

16-38. The purity of a chemical product is measured every two hours. The results of 20 consecutive measurements are as follows:

Sample	Purity	Sample	Purity
1	89.11	11	88.55
2	90.59	12	90.43
3	91.03	13	91.04
4	89.46	14	88.17
5	89.78	15	91.23
6	90.05	16	90.92
7	90.63	17	88.86
8	90.75	18	90.87
9	89.65	19	90.73
10	90.15	20	89.78

(a) Set up a CUSUM control chart for this process. Use $\sigma = 0.8$ in setting up the procedure, and assume that the desired process target is 90. Does the process appear to be in control?

(b) Suppose that the next five observations are 90.75, 90.00, 91.15, 90.95, and 90.86. Apply the CUSUM in part (a) to these new observations. Is there any evidence that the process has shifted out of control?

16-39. The diameter of holes is measured in consecutive order by an automatic sensor. The results of measuring 25 holes follow.

(a) Estimate the process standard deviation.

(b) Set up a CUSUM control procedure, assuming that the target diameter is 10.0 millimeters. Does the process

Sample	Diameter	Sample	Diameter
1	9.94	14	9.99
2	9.93	15	10.12
3	10.09	16	9.81
4	9.98	17	9.73
5	10.11	18	10.14
6	9.99	19	9.96
7	10.11	20	10.06
8	9.84	21	10.11
9	9.82	22	9.95
10	10.38	23	9.92
11	9.99	24	10.09
12	10.41	25	9.85
13	10.36		

appear to be operating in a state of statistical control at the desired target level?

16-40. The concentration of a chemical product is measured by taking four samples from each batch of material. The average concentration of these measurements is shown for the last 20 batches in the following table:

Batch	Concentration	Batch	Concentration
1	104.5	11	95.4
2	99.9	12	94.5
3	106.7	13	104.5
4	105.2	14	99.7
5	94.8	15	97.7
6	94.6	16	97.0
7	104.4	17	95.8
8	99.4	18	97.4
9	100.3	19	99.0
10	100.3	20	102.6

(a) Suppose that the process standard deviation is $\sigma = 8$ and that the target value of concentration for this process is 100. Design a CUSUM scheme for the process. Does the process appear to be in control at the target?

(b) How many batches would you expect to be produced with off-target concentration before it would be detected by the CUSUM control chart if the concentration shifted to 104? Use Table 16-9.

16-41. Consider a standardized CUSUM with $H = 5$ and $K = 1/2$. Samples are taken every two hours from the process. The target value for the process is $\mu_0 = 50$ and $\sigma = 2$. Use Table 16-9.

(a) If the sample size is $n = 1$, how many samples would be required to detect a shift in the process mean to $\mu = 51$ on average?

(b) If the sample size is increased to $n = 4$, how does this affect the average run length to detect the shift to $\mu = 51$ that you determined in part (a)?

16-42. A process has a target of $\mu_0 = 100$ and a standard deviation of $\sigma = 4$. Samples of size $n = 1$ are taken every two hours. Use Table 16-9.

(a) Suppose the process mean shifts to $\mu = 102$. How many hours of production will occur before the process shift is detected by a standardized CUSUM with $H = 5$ and $K = 1/2$?

(b) It is important to detect the shift defined in part (a) more quickly. A proposal is made to reduce the sampling frequency to 0.5 hour. How will this affect the CUSUM control procedure? How much more quickly will the shift be detected?

(c) Suppose that the 0.5 hour sampling interval in part (b) is adopted. How often will false alarms occur with this new sampling interval? How often did they occur with the old interval of two hours?

(d) A proposal is made to increase the sample size to $n = 4$ and retain the two-hour sampling interval. How does this suggestion compare in terms of average detection time to the suggestion of decreasing the sampling interval to 0.5 hour?

16-11 OTHER SPC PROBLEM-SOLVING TOOLS

While the control chart is a very powerful tool for investigating the causes of variation in a process, it is most effective when used with other SPC problem-solving tools. In this section we illustrate some of these tools, using the printed circuit board defect data in Example 16-4.

Figure 16-17 shows a U chart for the number of defects in samples of five printed circuit boards. The chart exhibits statistical control, but the number of defects must be reduced. The average number of defects per board is $8/5 = 1.6$, and this level of defects would require extensive rework.

The first step in solving this problem is to construct a **Pareto diagram** of the individual defect types. The Pareto diagram, shown in Fig. 16-22, indicates that insufficient solder and solder balls are the most frequently occurring defects, accounting for $(109/160)\ 100 = 68\%$ of the

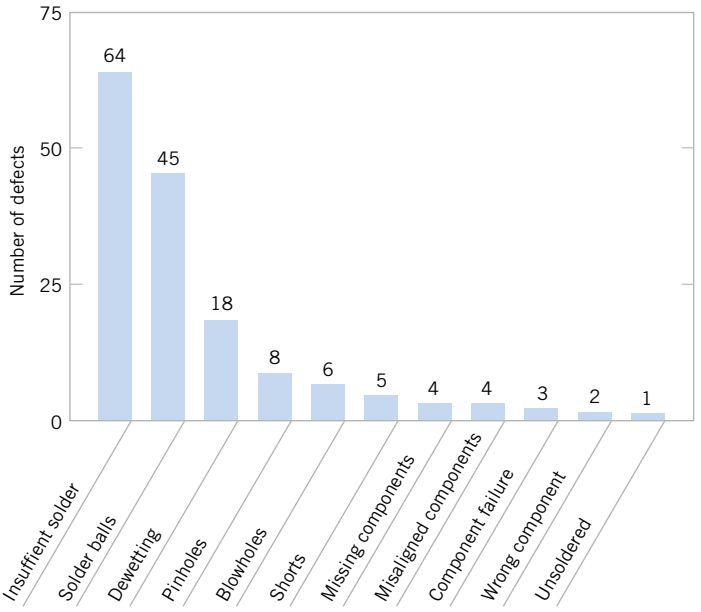

Figure 16-22 Pareto diagram for printed circuit board defects.

observed defects. Furthermore, the first five defect categories on the Pareto chart are all solder-related defects. This points to the flow solder process as a potential opportunity for improvement.

To improve the flow solder process, a team consisting of the flow solder operator, the shop supervisor, the manufacturing engineer responsible for the process, and a quality engineer meets to study potential causes of solder defects. They conduct a brainstorming session and produce the cause-and-effect diagram shown in Fig. 16-23. The cause-and-effect diagram is widely used to display the various potential causes of defects in products and their interrelationships. They are useful in summarizing knowledge about the process.

As a result of the brainstorming session, the team tentatively identifies the following variables as potentially influential in creating solder defects:

1. Flux specific gravity
2. Solder temperature

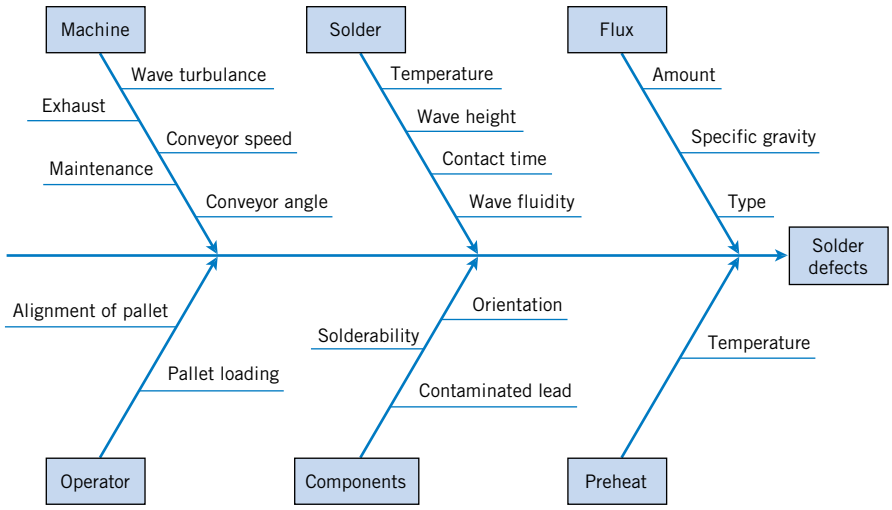

Figure 16-23 Cause-and-effect diagram for the printed circuit board flow solder process.

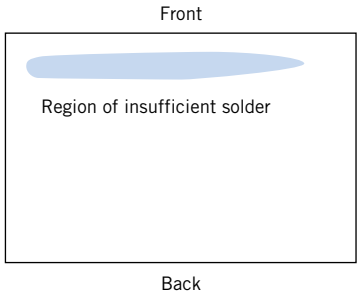

Figure 16-24 Defect concentration diagram for a printed circuit board.

3. Conveyor speed
4. Conveyor angle
5. Solder wave height
6. Preheat temperature
7. Pallet loading method

A statistically **designed experiment** could be used to investigate the effect of these seven variables on solder defects.

In addition, the team constructed a **defect concentration diagram** for the product. A defect concentration diagram is just a sketch or drawing of the product, with the most frequently occurring defects shown on the part. This diagram is used to determine whether defects occur in the same location on the part. The defect concentration diagram for the printed circuit board is shown in Fig. 16-24. This diagram indicates that most of the insufficient solder defects are near the front edge of the board, where it makes initial contact with the solder wave. Further investigation showed that one of the pallets used to carry the boards across the wave was bent, causing the front edge of the board to make poor contact with the solder wave.

When the defective pallet was replaced, a designed experiment was used to investigate the seven variables discussed earlier. The results of this experiment indicated that several of these factors were influential and could be adjusted to reduce solder defects. After the results of the experiment were implemented, the percentage of solder joints requiring rework was reduced from 1% to under 100 parts per million (0.01%).

16-12 IMPLEMENTING SPC

The methods of statistical process control can provide significant payback to those companies that can successfully implement them. While SPC seems to be a collection of statistically based problem-solving tools, there is more to the successful use of SPC than simply learning and using these tools. Management involvement and commitment to the quality-improvement process is the most vital component of SPC's potential success. Management is a role model, and others in the organization will look to management for guidance and as an example. A team approach is also important, for it is usually difficult for one person alone to introduce process improvements. Many of the "magnificent seven" problem-solving tools are helpful in building an improvement team, including cause-and-effect diagrams, Pareto charts, and defect concentration diagrams. The basic SPC problem-solving tools must become widely known and widely used throughout the organization. Continuous training in SPC and quality improvement is necessary to achieve this widespread knowledge of the tools.

The objective of an SPC-based quality-improvement program is continuous improvement on a weekly, quarterly, and annual basis. SPC is not a one-time program to be applied

when the business is in trouble and later abandoned. Quality improvement must become part of the culture of the organization.

The control chart is an important tool for process improvement. Processes do not naturally operate in an in-control state, and the use of control charts is an important step that must be taken early in an SPC program to eliminate assignable causes, reduce process variability, and stabilize process performance. To improve quality and productivity, we must begin to manage with facts and data, and not just rely on judgment. Control charts are an important part of this change in management approach.

In implementing a company-wide SPC program, we have found that the following elements are usually present in all successful efforts:

1. Management leadership
2. A team approach
3. Education of employees at all levels
4. Emphasis on continuous improvement
5. A mechanism for recognizing success

We cannot overemphasize the importance of management leadership and the team approach. Successful quality improvement is a "top-down" management-driven activity. It is also important to measure progress and success and to spread knowledge of this success throughout the organization. When successful improvements are communicated throughout the company, this can provide motivation and incentive to improve other processes and to make continuous improvement a normal part of the way of doing business.

The philosophy of W. Edwards **Deming** provides an important framework for implementing quality and productivity improvement. Deming's philosophy is summarized in his 14 points for management. The adherence to these management principles has been an important factor in Japan's industrial success and continues to be the catalyst in that nation's quality- and productivity-improvement efforts. This philosophy has also now spread rapidly in the West. **Deming's 14 points** are as follows.

1. **Create a constancy of purpose focused on the improvement of products and services.** Constantly try to improve product design and performance. Investment in research, development, and innovation will have a long-term payback to the organization.
2. **Adopt a new philosophy of rejecting poor workmanship, defective products, or bad service.** It costs as much to produce a defective unit as it does to produce a good one (and sometimes more). The cost of dealing with scrap, rework, and other losses created by defectives is an enormous drain on company resources.
3. **Do not rely on mass inspection to "control" quality.** All inspection can do is sort out defectives, and at this point it is too late because we have already paid to produce these defectives. Inspection occurs too late in the process, it is expensive, and it is often ineffective. Quality results from the prevention of defectives through process improvement, not inspection.
4. **Do not award business to suppliers on the basis of price alone, but also consider quality.** Price is a meaningful measure of a supplier's product only if it is considered in relation to a measure of quality. In other words, the total cost of the item must be considered, not just the purchase price. When quality is considered, the lowest bidder is frequently not the low-cost supplier. Preference should be given to suppliers who use modern methods of quality improvement in their business and who can demonstrate process control and capability.

5. **Focus on continuous improvement.** Constantly try to improve the production and service system. Involve the workforce in these activities and make use of statistical methods, particularly the SPC problem-solving tools discussed in the previous section.
6. **Practice modern training methods and invest in training for all employees.** Everyone should be trained in the technical aspects of their job, as well as in modern quality- and productivity-improvement methods. The training should encourage all employees to practice these methods every day.
7. **Practice modern supervision methods.** Supervision should not consist merely of passive surveillance of workers, but should be focused on helping the employees improve the system in which they work. The first goal of supervision should be to improve the work system and the product.
8. **Drive out fear.** Many workers are afraid to ask questions, report problems, or point out conditions that are barriers to quality and effective production. In many organizations the economic loss associated with fear is large; only management can eliminate fear.
9. **Break down the barriers between functional areas of the business.** Teamwork among different organizational units is essential for effective quality and productivity improvement to take place.
10. **Eliminate targets, slogans, and numerical goals for the workforce.** A target such as "zero defects" is useless without a plan as to how to achieve this objective. In fact, these slogans and "programs" are usually counterproductive. Work to improve the system and provide information on that.
11. **Eliminate numerical quotas and work standards.** These standards have historically been set without regard to quality. Work standards are often symptoms of management's inability to understand the work process and to provide an effective management system focused on improving this process.
12. **Remove the barriers that discourage employees from doing their jobs.** Management must listen to employee suggestions, comments, and complaints. The person who is doing the job is the one who knows the most about it, and usually has valuable ideas about how to make the process work more effectively. The workforce is an important participant in the business, and not just an opponent in collective bargaining.
13. **Institute an ongoing program of training and education for all employees.** Education in simple, powerful statistical techniques should be mandatory for all employees. Use of the basic SPC problem-solving tools, particularly the control chart, should become widespread in the business. As these charts become widespread, and as employees understand their uses, they will be more likely to look for the causes of poor quality and to identify process improvements. Education is a way of making everyone partners in the quality-improvement process.
14. **Create a structure in top management that will vigorously advocate the first 13 points.**

As we read Deming's 14 points, we notice two things. First, there is a strong emphasis on change. Second, the role of management in guiding this change process is of dominating importance. But what should be changed, and how should this change process be started? For example, if we want to improve the yield of a semiconductor manufacturing process, what should we do? It is in this area that statistical methods most frequently come into play. To improve the semiconductor process, we must determine which controllable factors in the process

influence the number of defective units produced. To answer this question, we must collect data on the process and see how the system reacts to changes in the process variables. Statistical methods, including the SPC and experimental design techniques in this book, can contribute to this knowledge.

SUPPLEMENTAL EXERCISES

16-43. The diameter of fuse pins used in an aircraft engine application is an important quality characteristic. Twenty-five samples of three pins each are shown as follows:

Sample Number	Diameter		
1	64.030	64.002	64.019
2	63.995	63.992	64.001
3	63.988	64.024	64.021
4	64.002	63.996	63.993
5	63.992	64.007	64.015
6	64.009	63.994	63.997
7	63.995	64.006	63.994
8	63.985	64.003	63.993
9	64.008	63.995	64.009
10	63.998	74.000	63.990
11	63.994	63.998	63.994
12	64.004	64.000	64.007
13	63.983	64.002	63.998
14	64.006	63.967	63.994
15	64.012	64.014	63.998
16	64.000	63.984	64.005
17	63.994	64.012	63.986
18	64.006	64.010	64.018
19	63.984	64.002	64.003
20	64.000	64.010	64.013
21	63.988	64.001	64.009
22	64.004	63.999	63.990
23	64.010	63.989	63.990
24	64.015	64.008	63.993
25	63.982	63.984	63.995

(a) Set up $\overline{X}$ and R charts for this process. If necessary, revise limits so that no observations are out-of-control.
(b) Estimate the process mean and standard deviation.
(c) Suppose the process specifications are at 64 ± 0.02. Calculate an estimate of PCR. Does the process meet a minimum capability level of $PCR \geq 1.33$?
(d) Calculate an estimate of PCR_k. Use this ratio to draw conclusions about process capability.
(e) To make this process a six-sigma process, the variance σ^2 would have to be decreased such that $PCR_k = 2.0$. What should this new variance value be?
(f) Suppose the mean shifts to 64.01. What is the probability that this shift will be detected on the next sample? What is the ARL after the shift?

16-44. Rework Exercise 16-43 with $\overline{X}$ and S charts.

16-45. Plastic bottles for liquid laundry detergent are formed by blow molding. Twenty samples of $n = 100$ bottles are inspected in time order of production, and the fraction defective in each sample is reported. The data are as follows:

Sample	Fraction Defective
1	0.12
2	0.15
3	0.18
4	0.10
5	0.12
6	0.11
7	0.05
8	0.09
9	0.13
10	0.13
11	0.10
12	0.07
13	0.12
14	0.08
15	0.09
16	0.15
17	0.10
18	0.06
19	0.12
20	0.13

(a) Set up a P chart for this process. Is the process in statistical control?
(b) Suppose that instead of $n = 100$, $n = 200$. Use the data given to set up a P chart for this process. Revise the control limits if necessary.
(c) Compare your control limits for the P charts in parts (a) and (b). Explain why they differ. Also, explain why your assessment about statistical control differs for the two sizes of n.

16-46. Cover cases for a personal computer are manufactured by injection molding. Samples of five cases are taken from the process periodically, and the number of defects is noted. Twenty-five samples follow:

Sample	No. of Defects	Sample	No. of Defects
1	3	14	8
2	2	15	0
3	0	16	2
4	1	17	4
5	4	18	3
6	3	19	5
7	2	20	0
8	4	21	2
9	1	22	1
10	0	23	9
11	2	24	3
12	3	25	2
13	2		

(a) Using all the data, find trial control limits for this U chart for the process.
(b) Use the trial control limits from part (a) to identify out-of-control points. If necessary, revise your control limits.
(c) Suppose that instead of samples of 5 cases, the sample size was 10. Repeat parts (a) and (b). Explain how this change alters your answers to parts (a) and (b).

16-47. Consider the data in Exercise 16-46.

(a) Using all the data, find trial control limits for a C chart for this process.
(b) Use the trial control limits of part (a) to identify out-of-control points. If necessary, revise your control limits.
(c) Suppose that instead of samples of 5 cases, the sample was 10 cases. Repeat parts (a) and (b). Explain how this alters your answers to parts (a) and (b).

16-48. Suppose that a process is in control and an $\overline{X}$ chart is used with a sample size of 4 to monitor the process. Suddenly there is a mean shift of 1.5σ.
(a) If 3-sigma control limits are in use on the $\overline{X}$ chart, what is the probability that this shift will remain undetected for three consecutive samples?
(b) If 2-sigma control limits are in use on the $\overline{X}$ chart, what is the probability that this shift will remain undetected for three consecutive samples?
(c) Compare your answers to parts (a) and (b) and explain why they differ. Also, which limits you would recommend using and why?

16-49. Consider the control chart for individuals with 3-sigma limits.
(a) Suppose that a shift in the process mean of magnitude σ occurs. Verify that the ARL for detecting the shift is ARL = 43.9.
(b) Find the ARL for detecting a shift of magnitude 2σ in the process mean.
(c) Find the ARL for detecting a shift of magnitude 3σ in the process mean.
(d) Compare your answers to parts (a), (b), and (c) and explain why the ARL for detection is decreasing as the magnitude of the shift increases.

16-50. Consider a control chart for individuals, applied to a continuous 24-hour chemical process with observations taken every hour.
(a) If the chart has 3-sigma limits, verify that the in-control ARL is ARL = 370. How many false alarms would occur each 30-day month, on the average, with this chart?
(b) Suppose that the chart has 2-sigma limits. Does this reduce the ARL for detecting a shift in the mean of magnitude σ? (Recall that the ARL for detecting this shift with 3-sigma limits is 43.9.)
(c) Find the in-control ARL if 2-sigma limits are used on the chart. How many false alarms would occur each month with this chart? Is this in-control ARL performance satisfactory? Explain your answer.

16-51. The depth of a keyway is an important part quality characteristic. Samples of size $n = 5$ are taken every four hours from the process and 20 samples are summarized as follows:

Sample	$\overline{X}$	r
1	139.7	1.1
2	139.8	1.4
3	140.0	1.3
4	140.1	1.6
5	139.8	0.9
6	139.9	1.0
7	139.7	1.4
8	140.2	1.2
9	139.3	1.1
10	140.7	1.0
11	138.4	0.8
12	138.5	0.9
13	137.9	1.2
14	138.5	1.1
15	140.8	1.0
16	140.5	1.3
17	139.4	1.4
18	139.9	1.0
19	137.5	1.5
20	139.2	1.3

(a) Using all the data, find trial control limits for $\overline{X}$ and R charts. Is the process in control?
(b) Use the trial control limits from part (a) to identify out-of-control points. If necessary, revise your control limits. Then, estimate the process standard deviation.
(c) Suppose that the specifications are at 140 ± 2. Using the results from part (b), what statements can you make about process capability? Compute estimates of the appropriate process capability ratios.
(d) To make this process a "6-sigma process," the variance σ^2 would have to be decreased such that $PCR_k = 2.0$. What should this new variance value be?
(e) Suppose the mean shifts to 139.7. What is the probability that this shift will be detected on the next sample? What is the ARL after the shift?

16-52. A process is controlled by a *P* chart using samples of size 100. The center line on the chart is 0.05.
(a) What is the probability that the control chart detects a shift to 0.08 on the first sample following the shift?
(b) What is the probability that the control chart does not detect a shift to 0.07 on the first sample following the shift but does detect it on the second sample?
(c) Suppose that instead of a shift in the mean to 0.07, the mean shifts to 0.10. Repeat parts (a) and (b).
(d) Compare your answers for a shift to 0.07 and for a shift to 0.10. Explain why they differ. Also, explain why a shift to 0.10 is easier to detect.

16-53. Suppose the average number of defects in a unit is known to be 8. If the mean number of defects in a unit shifts to 16, what is the probability that it will be detected by the *U* chart on the first sample following the shift
(a) if the sample size is $n = 4$?
(b) if the sample size is $n = 10$?
Use a normal approximation for *U*.

16-54. Suppose the average number of defects in a unit is known to be 10. If the mean number of defects in a unit shifts to 14, what is the probability that it will be detected by the *U* chart on the first sample following the shift
(a) if the sample size is $n = 1$?
(b) if the sample size is $n = 4$?
Use a normal approximation for *U*.

16-55. Suppose that an $\overline{X}$ control chart with 2-sigma limits is used to control a process. Find the probability that a false out-of-control signal will be produced on the next sample. Compare this with the corresponding probability for the chart with 3-sigma limits and discuss. Comment on when you would prefer to use 2-sigma limits instead of 3-sigma limits.

16-56. Consider the $\overline{X}$ control chart with 2-sigma limits in Exercise 16-50.
(a) Find the probability of no signal on the first sample but a signal on the second.
(b) What is the probability that there will not be a signal in three samples?

16-57. Suppose a process has a $PCR = 2$, but the mean is exactly three standard deviations above the upper specification limit. What is the probability of making a product outside the specification limits?

16-58. Consider the hardness measurement data in Exercise 16-9. Set up a CUSUM scheme for this process using $\mu = 50$ and $\sigma = 2$, so that $K = 1$ and $H = 10$. Is the process in control?

16-59. Consider the data in Exercise 16-10. Set up a CUSUM scheme for this process assuming that $\mu = 80$ is the process target. Explain how you determined your estimate of σ and the CUSUM parameters *K* and *H*.

16-60. Reconsider the data in Exercise 16-12. Construct a CUSUM control chart for this process using $\mu_0 = 500$ as the process target. Explain how you determined your estimate of σ and the CUSUM parameters *H* and *K*.

MIND-EXPANDING EXERCISES

16-61. Suppose a process is in control, and 3-sigma control limits are in use on the $\overline{X}$ chart. Let the mean shift by 1.5σ. What is the probability that this shift will remain undetected for three consecutive samples? What would its probability be if 2-sigma control limits were used? The sample size is 4.

16-62. Consider an $\overline{X}$ control chart with *k*-sigma control limits. Develop a general expression for the probability that a point will plot outside the control limits when the process mean has shifted by δ units from the center line.

16-63. Suppose that an $\overline{X}$ chart is used to control a normally distributed process and that samples of size *n* are taken every *n* hours and plotted on the chart, which has *k*-sigma limits.
(a) Find a general expression for the expected number of samples and time that will be taken until a false action signal is generated.
(b) Suppose that the process mean shifts to an out-of-control state, say $\mu_1 = \mu_0 + \delta\sigma$. Find an expression for the expected number of samples that will be taken until a false action is generated.

MIND-EXPANDING EXERCISES

(c) Evaluate the in-control ARL for $k = 3$. How, does this change if $k = 2$? What do you think about the use of 2-sigma limits in practice?
(d) Evaluate the out-of-control ARL for a shift of 1 sigma, given that $n = 5$.

16-64. Suppose a P chart with center line at $\bar{p}$ with k-sigma control limits is used to control a process. There is a critical fraction defective p_c that must be detected with probability 0.50 on the first sample following the shift to this state. Derive a general formula for the sample size that should be used on this chart.

16-65. Suppose that a P chart with center line at $\bar{p}$ and k-sigma control limits is used to control a process. What is the smallest sample size that can be used on this control chart to ensure that the lower control limit is positive?

16-66. A process is controlled by a P chart using samples of size 100. The center line on the chart is 0.05. What is the probability that the control chart detects a shift to 0.08 on the first sample following the shift? What is the probability that the shift is detected by at least the third sample following the shift?

16-67. Consider a process where specifications on a quality characteristic are 100 ± 15. We know that the standard deviation of this normally distributed quality characteristic is 5. Where should we center the process to minimize the fraction defective produced? Now suppose the mean shifts to 105 and we are using a sample size of 4 on an $\bar{X}$ chart. What is the probability that such a shift will be detected on the first sample following the shift? What is the average number of samples until an out-of-control point occurs? Compare this result to the average number of observations until a defective occurs (assuming normality).

16-68. **The *NP* Control Chart.** An alternative to the control chart for fraction defective is a control chart based on the number of defectives, or the NP control chart. The chart has centerline at $n\bar{p}$, and the control limits are

$$UCL = n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})}$$

$$LCL = n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})}$$

and the number of defectives for each sample is plotted on the chart.
(a) Verify that the control limits given above are correct.
(b) Apply this control chart to the data in Example 16-4.
(c) Will this chart always provide results that are equivalent to the usual P chart?

16-69. **The EWMA Control Chart.** The exponentially weighted moving average (or EWMA) is defined as follows:

$$Z_t = \lambda \bar{X}_t + (1-\lambda)\bar{Z}_{t-1}$$

where $0 < \lambda \leq 1$, and the starting value of the EWMA at time $t = 0$ is $Z_0 = \mu_0$ (the process target). An EWMA control chart is constructed by plotting the Z_t values on a chart with center line at μ_0 and appropriate control limits.
(a) Verify that $E(Z_t) = \mu_0$
(b) Let $\sigma^2_{z_t}$ be $V(Z_t)$, and show that

$$\sigma^2_{z_t} = \frac{\sigma^2}{n}\left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2t}]$$

(c) Use the results of part (b) to determine the control limits for the EWMA chart.
(d) As $\lambda \to 1$, the EWMA control chart should perform like a standard Shewhart $\bar{X}$ chart. Do you agree with this statement? Why?
(e) As $\lambda \to 0$, the EWMA control chart should perform like a CUSUM. Provide an argument as to why this is so.
(f) Apply this procedure to the data in Example 16-2.

16-70. **Standardized Control Chart.** Consider the P chart with the usual 3-sigma control limits. Suppose that we define a new variable:

$$Z_i = \frac{\hat{P}_i - \bar{P}}{\sqrt{\dfrac{\bar{P}(1-\bar{P})}{n}}}$$

as the quantity to plot on a control chart. It is proposed that this new chart will have a center line at 0 with the upper and lower control limits at ± 3. Verify that this standardized control chart will be equivalent to the original p chart.

16-71. **Unequal Sample Sizes.** One application of the standardized control chart introduced in Exercise 16-70 is to allow unequal sample sizes on the control chart. Provide details concerning how this procedure would be implemented and illustrate using the following data:

Sample, i	1	2	3	4	5	6	7	8	9	10
n_i	20	25	20	25	50	30	25	25	25	20
p_i	0.2	0.16	0.25	0.08	0.3	0.1	0.12	0.16	0.12	0.15

IMPORTANT TERMS AND CONCEPTS

In the E-book, click on any term or concept below to go to that subject.

ARL
Assignable causes
Attributes control charts
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